

THE NUCLEAR SCISSORS MODE WITHIN TWO APPROACHES (Wigner function moments versus RPA)

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Abstract

Two complementary methods to describe the collective motion, RPA and Wigner function moments method, are compared on an example of a simple model – harmonic oscillator with quadrupole–quadrupole residual interaction. It is shown that they give identical formulae for eigenfrequencies and transition probabilities of all collective excitations of the model including the scissors mode, which here is the subject of our special attention. The exact relation between the variables of the two methods and the respective dynamical equations is established. The normalization factor of the “synthetic” scissors state and its overlap with physical states are calculated analytically. The orthogonality of the spurious state to all physical states is proved rigorously.

1 Introduction

The full analysis of the scissors mode in the framework of a solvable model (harmonic oscillator with quadrupole–quadrupole residual interaction (HO+QQ)) was given in [1]. Several points in the understanding of the nature of this mode were clarified: for example, its coexistence with the isovector giant quadrupole resonance (IVGQR), the decisive role of the Fermi surface deformation, and several things more.

The Wigner Function Moments (WFM) method was applied to derive analytical expressions for currents of both coexisting modes (for the first time), their excitation energies, magnetic and electric transition probabilities. Our formulae for energies turned out to be identical with those derived by Hamamoto and Nazarewicz [2] in the framework of the RPA. This fact generated the natural motivation for this work: to investigate the relation between formulas for transition probabilities derived by two methods. More generally we will perform a systematic comparison of the two approaches. The HO+QQ model is a very convenient ground for this kind of research, because all results can be obtained analytically. There is no need to describe the merits of the RPA – they are very well known [3]. It is necessary, however, to say a few words about the WFM. Its idea is based on the virial theorems of Chandrasekhar and Lebovitz [4]. Instead of writing the equations of motion for microscopic amplitudes of particle–hole excitations (RPA), one writes the dynamical equations for various multipole phase space moments of a nucleus. This allows one to achieve a better physical interpretation of the studied phenomenon without going into its detailed microscopic structure. The WFM method was successfully applied to the study of isoscalar and isovector giant multipole resonances and low-lying collective modes of rotating and nonrotating nuclei with various realistic forces [5]. The results of WFM were always very close to similar results obtained with the help of RPA. In principle, this should be expected, because the basis of both the methods is the same: Time Dependent Hartree–Fock (TDHF) theory with its small amplitude approximation. On the other hand, they are equivalent only in special cases. The detailed analysis of the interplay of the two methods turns out to be useful also from a “practical” point of view: firstly, it allows one to obtain additional insight into the nature of the scissors mode; secondly, we find new exact mathematical results for the considered model.

The paper is organized as follows. In Section 2 we remind the principal points of the

WFM formalism and give the summary of the key results of [1] obtained by applying this method to the HO+QQ model. The same model is considered in Section 3 in the frame of RPA: the formulae for eigenfrequencies, electric and magnetic transition probabilities of the scissors mode are derived, the “synthetic” scissors and spurious state are analyzed, the RPA equations of motion for transition matrix elements are compared with the WFM equations of motion for irreducible tensors. The exact relation between the RPA and WFM variables is established in Section 4. The mutual interplay of the two methods is discussed in the conclusion. The various mathematical details are given in Appendices A and B.

2 The WFM method

The basis of the method is the Time Dependent Hartree-Fock (TDHF) equation for the one-body density matrix $\rho^\tau(\mathbf{r}_1, \mathbf{r}_2, t) = \langle \mathbf{r}_1 | \hat{\rho}^\tau(t) | \mathbf{r}_2 \rangle$:

$$i\hbar \frac{\partial \hat{\rho}^\tau}{\partial t} = [\hat{H}^\tau, \hat{\rho}^\tau], \quad (1)$$

where \hat{H}^τ is the one-body self-consistent mean field Hamiltonian depending implicitly on the density matrix and τ is an isotopic spin index. It is convenient to modify equation (1) introducing the Wigner transform of the density matrix

$$f^\tau(\mathbf{r}, \mathbf{p}, t) = \int d^3s \exp(-i\mathbf{p} \cdot \mathbf{s}/\hbar) \rho^\tau(\mathbf{r} + \frac{\mathbf{s}}{2}, \mathbf{r} - \frac{\mathbf{s}}{2}, t) \quad (2)$$

and of the Hamiltonian

$$H_W^\tau(\mathbf{r}, \mathbf{p}) = \int d^3s \exp(-i\mathbf{p} \cdot \mathbf{s}/\hbar) (\mathbf{r} + \frac{\mathbf{s}}{2} | \hat{H}^\tau | \mathbf{r} - \frac{\mathbf{s}}{2}). \quad (3)$$

Using (2,3) one arrives [3] at

$$\frac{\partial f^\tau}{\partial t} = \frac{2}{\hbar} \sin \left\{ \frac{\hbar}{2} [(\nabla)^H \cdot (\nabla^p)^f - (\nabla^p)^H \cdot (\nabla)^f] \right\} H_W^\tau f^\tau, \quad (4)$$

where the upper index on the bracket stands for the function on which the operator in these brackets acts. It is shown in [5, 6], that by integrating equation (4) over the phase space $\{\mathbf{p}, \mathbf{r}\}$ with the weights $x_{i_1} x_{i_2} \dots x_{i_k} p_{i_{k+1}} \dots p_{i_{n-1}} p_{i_n}$, where k runs from 0 to n , one can obtain a closed finite set of dynamical equations for Cartesian tensors of the rank

n . Taking linear combinations of these equations one is able to represent them through irreducible tensors, which play the role of collective variables of the problem. However, it is more convenient to derive the dynamical equations directly for irreducible tensors using the technique of tensor products [7]. For this it is necessary to rewrite the Wigner function equation (4) in terms of cyclic variables

$$\frac{\partial f^\tau}{\partial t} = \frac{2}{\hbar} \sin \left\{ \frac{\hbar}{2} \sum_{\alpha=-1}^1 (-1)^\alpha \left[(\nabla_{-\alpha})^H \cdot (\nabla_\alpha^p)^f - (\nabla_{-\alpha}^p)^H \cdot (\nabla_\alpha)^f \right] \right\} H_W^\tau f^\tau, \quad (5)$$

with

$$\begin{aligned} \nabla_{+1} &= -\frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} \right), \quad \nabla_0 = \frac{\partial}{\partial x_3}, \quad \nabla_{-1} = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \right), \\ r_{+1} &= -\frac{1}{\sqrt{2}} (x_1 + ix_2), \quad r_0 = x_3, \quad r_{-1} = \frac{1}{\sqrt{2}} (x_1 - ix_2) \end{aligned}$$

and the analogous definitions for ∇_{+1}^p , ∇_0^p , ∇_{-1}^p , and p_{+1} , p_0 , p_{-1} . The required equations are obtained by integrating (5) with different tensor products of r_α and p_α . Here we consider the case $n = 2$.

2.1 Model Hamiltonian, Equations of motion

The microscopic Hamiltonian of the model is

$$\begin{aligned} H &= \sum_{i=1}^A \left(\frac{\hat{\mathbf{p}}_i^2}{2m} + \frac{1}{2} m \omega^2 \mathbf{r}_i^2 \right) + \bar{\kappa} \sum_{\mu=-2}^2 (-1)^\mu \sum_i^Z \sum_j^N q_{2-\mu}(\mathbf{r}_i) q_{2\mu}(\mathbf{r}_j) \\ &+ \frac{1}{2} \kappa \sum_{\mu=-2}^2 (-1)^\mu \left\{ \sum_{i \neq j}^Z q_{2-\mu}(\mathbf{r}_i) q_{2\mu}(\mathbf{r}_j) + \sum_{i \neq j}^N q_{2-\mu}(\mathbf{r}_i) q_{2\mu}(\mathbf{r}_j) \right\}, \end{aligned} \quad (6)$$

where the quadrupole operator $q_{2\mu} = \sqrt{16\pi/5} r^2 Y_{2\mu}$ and N, Z are the numbers of neutrons and protons, respectively. The mean field potential for protons (or neutrons) is

$$V^\tau(\mathbf{r}, t) = \frac{1}{2} m \omega^2 r^2 + \sum_{\mu=-2}^2 (-1)^\mu \tilde{Z}_{2-\mu}^\tau(t) q_{2\mu}(\mathbf{r}), \quad (7)$$

where $\tilde{Z}_{2\mu}^n = \kappa Q_{2\mu}^n + \bar{\kappa} Q_{2\mu}^p$, $\tilde{Z}_{2\mu}^p = \kappa Q_{2\mu}^p + \bar{\kappa} Q_{2\mu}^n$ and the quadrupole moments $Q_{2\mu}^\tau(t)$ are defined as

$$Q_{2\mu}^\tau(t) = \int d\{\mathbf{p}, \mathbf{r}\} q_{2\mu}(\mathbf{r}) f^\tau(\mathbf{r}, \mathbf{p}, t)$$

with $\int d\{\mathbf{p}, \mathbf{r}\} \equiv 2(2\pi\hbar)^{-3} \int d^3p \int d^3r$, where the factor 2 appears due to summation over spin degrees of freedom. To simplify notation we omit spin indices, because we consider spin saturated system without the spin-orbit interaction.

Substituting spherical functions by tensor products $r^2 Y_{2\mu} = \sqrt{\frac{15}{8\pi}} r_{2\mu}^2$, where

$$r_{\lambda\mu}^2 \equiv \{r \otimes r\}_{\lambda\mu} = \sum_{\sigma,\nu} C_{1\sigma,1\nu}^{\lambda\mu} r_{\sigma} r_{\nu}$$

and $C_{1\sigma,1\nu}^{\lambda\mu}$ is the Clebsch-Gordan coefficient, one has

$$V^{\tau} = \frac{1}{2} m \omega^2 r^2 + \sum_{\mu} (-1)^{\mu} Z_{2-\mu}^{\tau} r_{2\mu}^2. \quad (8)$$

Here

$$Z_{2\mu}^n = \chi R_{2\mu}^n + \bar{\chi} R_{2\mu}^p, \quad Z_{2\mu}^p = \chi R_{2\mu}^p + \bar{\chi} R_{2\mu}^n, \quad \chi = 6\kappa, \quad \bar{\chi} = 6\bar{\kappa},$$

$$R_{\lambda\mu}^{\tau}(t) = \int d\{\mathbf{p}, \mathbf{r}\} r_{\lambda\mu}^2 f^{\tau}(\mathbf{r}, \mathbf{p}, t). \quad (9)$$

Integration of equation (5) with the weights $r_{\lambda\mu}^2$, $(rp)_{\lambda\mu} \equiv \{r \otimes p\}_{\lambda\mu}$ and $p_{\lambda\mu}^2$ yields the following set of equations [1]:

$$\begin{aligned} \frac{d}{dt} R_{\lambda\mu}^{\tau} - \frac{2}{m} L_{\lambda\mu}^{\tau} &= 0, \quad \lambda = 0, 2 \\ \frac{d}{dt} L_{\lambda\mu}^{\tau} - \frac{1}{m} P_{\lambda\mu}^{\tau} + m \omega^2 R_{\lambda\mu}^{\tau} - 2\sqrt{5} \sum_{j=0}^2 \sqrt{2j+1} \{ \begin{smallmatrix} 11j \\ 2\lambda 1 \end{smallmatrix} \} (Z_2^{\tau} R_j^{\tau})_{\lambda\mu} &= 0, \quad \lambda = 0, 1, 2 \\ \frac{d}{dt} P_{\lambda\mu}^{\tau} + 2m \omega^2 L_{\lambda\mu}^{\tau} - 4\sqrt{5} \sum_{j=0}^2 \sqrt{2j+1} \{ \begin{smallmatrix} 11j \\ 2\lambda 1 \end{smallmatrix} \} (Z_2^{\tau} L_j^{\tau})_{\lambda\mu} &= 0, \quad \lambda = 0, 2 \end{aligned} \quad (10)$$

where $\{ \begin{smallmatrix} 11j \\ 2\lambda 1 \end{smallmatrix} \}$ is the Wigner 6j-symbol and the following notation is introduced

$$P_{\lambda\mu}^{\tau}(t) = \int d\{\mathbf{p}, \mathbf{r}\} p_{\lambda\mu}^2 f^{\tau}(\mathbf{r}, \mathbf{p}, t), \quad L_{\lambda\mu}^{\tau}(t) = \int d\{\mathbf{p}, \mathbf{r}\} (rp)_{\lambda\mu} f^{\tau}(\mathbf{r}, \mathbf{p}, t).$$

By definition $q_{2\mu} = \sqrt{6} r_{2\mu}^2$, $Q_{2\mu}^{\tau} = \sqrt{6} R_{2\mu}^{\tau}$, $R_{00}^{\tau} = -Q_{00}^{\tau}/\sqrt{3}$ with $Q_{00}^{\tau} = N_{\tau} < r^2 >$ being the mean square radius of neutrons or protons. The tensor $L_{1\nu}^{\tau}$ is connected with angular momentum by the relations $L_{10}^{\tau} = \frac{i}{\sqrt{2}} I_3^{\tau}$, $L_{1\pm 1}^{\tau} = \frac{1}{2} (I_2^{\tau} \mp i I_1^{\tau})$.

We rewrite equations (10) in terms of the isoscalar and isovector variables $R_{\lambda\mu} = R_{\lambda\mu}^n + R_{\lambda\mu}^p$, $\bar{R}_{\lambda\mu} = R_{\lambda\mu}^n - R_{\lambda\mu}^p$ (and so on) with the isoscalar $\kappa_0 = (\kappa + \bar{\kappa})/2$ and isovector $\kappa_1 = (\kappa - \bar{\kappa})/2$ strength constants. There is no problem to solve these equations numerically. However, we want to simplify the situation as much as possible to get the results in analytical form giving us a maximum of insight into the nature of the modes.

1) We consider the problem in small-amplitude approximation. Writing all variables as a sum of their equilibrium value plus a small deviation

$$R_{\lambda\mu}(t) = R_{\lambda\mu}^{eq} + \mathcal{R}_{\lambda\mu}(t), \quad P_{\lambda\mu}(t) = P_{\lambda\mu}^{eq} + \mathcal{P}_{\lambda\mu}(t), \quad L_{\lambda\mu}(t) = L_{\lambda\mu}^{eq} + \mathcal{L}_{\lambda\mu}(t),$$

$$\bar{R}_{\lambda\mu}(t) = \bar{R}_{\lambda\mu}^{eq} + \bar{\mathcal{R}}_{\lambda\mu}(t), \quad \bar{P}_{\lambda\mu}(t) = \bar{P}_{\lambda\mu}^{eq} + \bar{\mathcal{P}}_{\lambda\mu}(t), \quad \bar{L}_{\lambda\mu}(t) = \bar{L}_{\lambda\mu}^{eq} + \bar{\mathcal{L}}_{\lambda\mu}(t),$$

we linearize the equations of motion in $\mathcal{R}_{\lambda\mu}$, $\mathcal{P}_{\lambda\mu}$, $\mathcal{L}_{\lambda\mu}$ and $\bar{\mathcal{R}}_{\lambda\mu}$, $\bar{\mathcal{P}}_{\lambda\mu}$, $\bar{\mathcal{L}}_{\lambda\mu}$.

2) We study non-rotating nuclei, i.e. nuclei with $L_{1\nu}^{eq} = \bar{L}_{1\nu}^{eq} = 0$.

3) Only axially symmetric nuclei with $R_{2\pm 2}^{eq} = R_{2\pm 1}^{eq} = \bar{R}_{2\pm 2}^{eq} = \bar{R}_{2\pm 1}^{eq} = 0$ are considered.

4) Finally, we take

$$\bar{R}_{20}^{eq} = \bar{R}_{00}^{eq} = 0. \quad (11)$$

This means that equilibrium deformation and mean square radius of neutrons are supposed to be equal to that of protons.

Due to the approximation (11) the equations for isoscalar and isovector systems are decoupled. Further, due to the axial symmetry the angular momentum projection is a good quantum number. As a result, every set of equations splits into five independent subsets with $\mu = 0, \pm 1, \pm 2$. The detailed derivation of formulae for eigenfrequencies and transition probabilities together with all necessary explanations are given in [1]. Here we write out only the final results required for the comparison with respective results obtained in the framework of RPA.

2.2 Isoscalar eigenfrequencies

The isoscalar subset of equations with $\mu = 1$ is

$$\begin{aligned} \dot{\mathcal{R}}_{21} - 2\mathcal{L}_{21}/m &= 0, \\ \dot{\mathcal{L}}_{21} - \mathcal{P}_{21}/m + [m\omega^2 + 2\kappa_0(Q_{20}^{eq} + 2Q_{00}^{eq})] \mathcal{R}_{21} &= 0, \\ \dot{\mathcal{P}}_{21} + 2[m\omega^2 + \kappa_0 Q_{20}^{eq}] \mathcal{L}_{21} &= 0, \\ \dot{\mathcal{L}}_{11} &= 0. \end{aligned} \quad (12)$$

Imposing the time evolution via $e^{-i\Omega t}$ for all variables one transforms (12) into a set of algebraic equations. The eigenfrequencies are found from its characteristic equation which reads

$$\Omega^2[\Omega^2 - 4\omega^2 - \frac{6\kappa_0}{m}(Q_{20}^{eq} + \frac{4}{3}Q_{00}^{eq})] = 0. \quad (13)$$

For κ_0 we take the self-consistent value $\kappa_0 = -\frac{m\bar{\omega}^2}{4Q_{00}}$, where $\bar{\omega}^2 = \frac{\omega^2}{1 + \frac{2}{3}\delta}$ (see Appendix A)

with the standard definition of the deformation parameter $Q_{20} = Q_{00}\frac{4}{3}\delta$. Then

$$\Omega^2[\Omega^2 - 2\bar{\omega}^2(1 + \delta/3)] = 0. \quad (14)$$

The nontrivial solution of this equation gives the frequency of the $\mu = 1$ branch of the isoscalar GQR

$$\Omega^2 = \Omega_{is}^2 = 2\bar{\omega}^2(1 + \delta/3). \quad (15)$$

Taking into account the relation (A.7) we find that this result coincides with that of [9]. The trivial solution $\Omega = \Omega_0 = 0$ is characteristic of nonvibrational mode corresponding to the obvious integral of motion $\mathcal{L}_{11} = \text{const}$ responsible for the rotational degree of freedom. This is usually called the ‘spurious’ or ‘Goldstone’ mode.

2.3 Isovector eigenfrequencies

The information about the scissors mode is contained in the subset of isovector equations with $\mu = 1$

$$\begin{aligned} \dot{\bar{\mathcal{R}}}_{21} - 2\bar{\mathcal{L}}_{21}/m &= 0, \\ \dot{\bar{\mathcal{L}}}_{21} - \bar{\mathcal{P}}_{21}/m + [m\omega^2 + \kappa Q_{20}^{eq} + 4\kappa_1 Q_{00}^{eq}] \bar{\mathcal{R}}_{21} &= 0, \\ \dot{\bar{\mathcal{P}}}_{21} + 2[m\omega^2 + \kappa_0 Q_{20}^{eq}] \bar{\mathcal{L}}_{21} - 6\kappa_0 Q_{20}^{eq} \bar{\mathcal{L}}_{11} &= 0, \\ \dot{\bar{\mathcal{L}}}_{11} + 3\bar{\kappa} Q_{20}^{eq} \bar{\mathcal{R}}_{21} &= 0. \end{aligned} \quad (16)$$

Imposing the time evolution via $e^{-i\Omega t}$ one transforms (16) into a set of algebraic equations. Again the eigenfrequencies are found from the characteristic equation which reads

$$\Omega^4 - \Omega^2[4\omega^2 + \frac{8}{m}\kappa_1 Q_{00}^{eq} + \frac{2}{m}(\kappa_1 + 2\kappa_0)Q_{20}^{eq}] + \frac{36}{m^2}(\kappa_0 - \kappa_1)\kappa_0(Q_{20}^{eq})^2 = 0. \quad (17)$$

Supposing, as usual, the isovector constant κ_1 to be proportional to the isoscalar one, $\kappa_1 = \alpha\kappa_0$, and taking the self-consistent value for κ_0 , we finally obtain

$$\Omega^4 - 2\Omega^2\bar{\omega}^2(2 - \alpha)(1 + \delta/3) + 4\bar{\omega}^4(1 - \alpha)\delta^2 = 0. \quad (18)$$

The solutions of this equation are

$$\Omega_{\pm}^2 = \bar{\omega}^2(2 - \alpha)(1 + \delta/3) \pm \sqrt{\bar{\omega}^4(2 - \alpha)^2(1 + \delta/3)^2 - 4\bar{\omega}^4(1 - \alpha)\delta^2}. \quad (19)$$

The high-lying solution Ω_+ gives the frequency Ω_{iv} of the $\mu = 1$ branch of the isovector GQR. The low-lying solution Ω_- gives the frequency Ω_{sc} of the scissors mode.

We adjust α from the fact that the IVGQR is experimentally known to lie practically at twice the energy of the isoscalar GQR. In our model the experimental situation is satisfied by $\alpha = -2$. Then

$$\Omega_{iv}^2 = 4\bar{\omega}^2 \left(1 + \frac{\delta}{3} + \sqrt{\left(1 + \frac{\delta}{3}\right)^2 - \frac{3}{4}\delta^2} \right), \quad \Omega_{sc}^2 = 4\bar{\omega}^2 \left(1 + \frac{\delta}{3} - \sqrt{\left(1 + \frac{\delta}{3}\right)^2 - \frac{3}{4}\delta^2} \right). \quad (20)$$

2.4 Linear response and transition probabilities

A direct way of calculating the reduced transition probabilities is provided by the theory of the linear response of a system to a weak external field

$$\hat{F}(t) = \hat{F} \exp(-i\Omega t) + \hat{F}^\dagger \exp(i\Omega t),$$

where $\hat{F} = \sum_{s=1}^A \hat{f}_s$ is a one-body operator. A convenient form of the response theory is e.g. given by Lane [8] (see also section 4). The matrix elements of the operator \hat{F} obey the relation

$$| \langle \nu | \hat{F} | 0 \rangle |^2 = \hbar \lim_{\Omega \rightarrow \Omega_\nu} (\Omega - \Omega_\nu) \overline{\langle \psi | \hat{F} | \psi \rangle \exp(-i\Omega t)}, \quad (21)$$

where $|0\rangle$ and $|\nu\rangle$ are the stationary wave functions of the ground and unperturbed excited states; ψ is the perturbed wavefunction of the ground state, $\Omega_\nu = (E_\nu - E_0)/\hbar$ are the normal frequencies, the bar means averaging over a time interval much larger than $1/\Omega$, Ω being the frequency of the external field $\hat{F}(t)$.

Magnetic excitations

$$\hat{F} = \hat{F}_{1\mu}^p = \sum_{s=1}^Z \hat{f}_{1\mu}(s), \quad \hat{f}_{1\mu} = -i\nabla(rY_{1\mu}) \cdot [\mathbf{r} \times \nabla] \mu_N = \gamma(r\hat{p})_{1\mu}, \quad \mu_N = \frac{e\hbar}{2mc}. \quad (22)$$

$$\langle \psi | \hat{F}_{1\mu}^p | \psi \rangle = \gamma L_{1\mu}^p = \frac{\gamma}{2} (L_{1\mu} - \bar{L}_{1\mu}) = \frac{\gamma}{2} (\mathcal{L}_{1\mu} - \bar{\mathcal{L}}_{1\mu}), \quad \gamma = -i \frac{e}{2mc} \sqrt{\frac{3}{2\pi}}.$$

$$B(M1)_{sc} = 2 | \langle sc | \hat{F}_{11}^p | 0 \rangle |^2 = \frac{1 - \alpha}{4\pi} \frac{m\bar{\omega}^2}{\hbar} Q_{00} \delta^2 \frac{\Omega_{sc}^2 - 2(1 + \delta/3)\bar{\omega}^2}{\Omega_{sc}(\Omega_{sc}^2 - \Omega_{iv}^2)} \mu_N^2, \quad (23)$$

$$B(M1)_{iv} = 2 | \langle iv | \hat{F}_{11}^p | 0 \rangle |^2 = \frac{1 - \alpha}{4\pi} \frac{m\bar{\omega}^2}{\hbar} Q_{00} \delta^2 \frac{\Omega_{iv}^2 - 2(1 + \delta/3)\bar{\omega}^2}{\Omega_{iv}(\Omega_{iv}^2 - \Omega_{sc}^2)} \mu_N^2. \quad (24)$$

These two formulae can be joined into one expression by the simple transformation of the denominators. Really, we have from (19)

$$\begin{aligned} \pm(\Omega_{iv}^2 - \Omega_{sc}^2) &= \pm(\Omega_+^2 - \Omega_-^2) = \pm 2\sqrt{\bar{\omega}^4(2 - \alpha)^2(1 + \delta/3)^2 - 4\bar{\omega}^4(1 - \alpha)\delta^2} \\ &= 2\Omega_\pm^2 - 2\bar{\omega}^2(2 - \alpha)(1 + \delta/3) = 2\Omega_\pm^2 - (2 - \alpha)(\omega_x^2 + \omega_z^2). \end{aligned} \quad (25)$$

Using these relations in formulae (23) and (24), we obtain the expression for the $B(M1)$ values valid for both excitations

$$B(M1)_\nu = 2 | \langle \nu | \hat{F}_{11}^p | 0 \rangle |^2 = \frac{1 - \alpha}{8\pi} \frac{m\bar{\omega}^2}{\hbar} Q_{00} \delta^2 \frac{\Omega_\nu^2 - 2(1 + \delta/3)\bar{\omega}^2}{\Omega_\nu[\Omega_\nu^2 - \bar{\omega}^2(2 - \alpha)(1 + \delta/3)]} \mu_N^2. \quad (26)$$

Electric excitations

$$\hat{F} = \hat{F}_{2\mu}^p = \sum_{s=1}^Z \hat{f}_{2\mu}(s), \quad \hat{f}_{2\mu} = e r^2 Y_{2\mu} = \beta r_{2\mu}^2, \quad \beta = e \sqrt{\frac{15}{8\pi}}. \quad (27)$$

$$\langle \psi | \hat{F}_{2\mu}^p | \psi \rangle = \beta R_{2\mu}^p = \frac{1}{2} \beta (R_{2\mu} - \bar{R}_{2\mu}).$$

$$B(E2)_{sc} = 2 | \langle sc | \hat{F}_{21}^p | 0 \rangle |^2 = \frac{e^2 \hbar}{m} \frac{5}{8\pi} Q_{00} \frac{(1 + \delta/3) \Omega_{sc}^2 - 2(\bar{\omega}\delta)^2}{\Omega_{sc}(\Omega_{sc}^2 - \Omega_{iv}^2)}. \quad (28)$$

$$B(E2)_{iv} = 2 | \langle iv | \hat{F}_{21}^p | 0 \rangle |^2 = \frac{e^2 \hbar}{m} \frac{5}{8\pi} Q_{00} \frac{(1 + \delta/3) \Omega_{iv}^2 - 2(\bar{\omega}\delta)^2}{\Omega_{iv}(\Omega_{iv}^2 - \Omega_{sc}^2)}. \quad (29)$$

$$B(E2)_{is} = 2 | \langle is | \hat{F}_{21}^p | 0 \rangle |^2 = \frac{e^2 \hbar}{m} \frac{5}{8\pi} Q_{00} [(1 + \delta/3) \Omega_{is}^2 - 2(\bar{\omega}\delta)^2] / [\Omega_{is}]^3. \quad (30)$$

Using relations (25) in formulae (28) and (29) we obtain the expression for the $B(E2)$ values valid for all three excitations

$$B(E2)_\nu = 2 | \langle \nu | \hat{F}_{21}^p | 0 \rangle |^2 = \frac{e^2 \hbar}{m} \frac{5}{16\pi} Q_{00} \frac{(1 + \delta/3) \Omega_\nu^2 - 2(\bar{\omega}\delta)^2}{\Omega_\nu[\Omega_\nu^2 - \bar{\omega}^2(2 - \alpha)(1 + \delta/3)]}. \quad (31)$$

The isoscalar value (30) is obtained by assuming $\alpha = 1$.

3 Random Phase Approximation (RPA)

Standard RPA equations in the notation of [3] are

$$\begin{aligned} \sum_{n,j} \{ [\delta_{ij} \delta_{mn} (\epsilon_m - \epsilon_i) + \bar{v}_{mjn}] X_{nj} + \bar{v}_{mni} Y_{nj} \} &= \hbar \Omega X_{mi}, \\ \sum_{n,j} \{ \bar{v}_{ijmn} X_{nj} + [\delta_{ij} \delta_{mn} (\epsilon_m - \epsilon_i) + \bar{v}_{inmj}] Y_{nj} \} &= -\hbar \Omega Y_{mi}. \end{aligned} \quad (32)$$

According to the definition of the schematic model by [3], the matrix elements of the residual interaction corresponding to the Hamiltonian (6) are

$$\bar{v}_{mj in} = \kappa_{\tau\tau'} D_{im}^{\tau*} D_{jn}^{\tau'}$$

with $D_{im} \equiv \langle i | q_{21} | m \rangle$ and $\kappa_{nn} = \kappa_{pp} = \kappa$, $\kappa_{np} = \bar{\kappa}$. This interaction distinguishes between protons and neutrons, so we have to introduce the isospin indices τ, τ' into the set of RPA equations (32):

$$\begin{aligned} (\epsilon_m^\tau - \epsilon_i^\tau) X_{mi}^\tau + \sum_{n,j,\tau'} \kappa_{\tau\tau'} D_{im}^{\tau*} D_{jn}^{\tau'} X_{nj}^{\tau'} + \sum_{n,j,\tau'} \kappa_{\tau\tau'} D_{im}^{\tau*} D_{nj}^{\tau'} Y_{nj}^{\tau'} &= \hbar\Omega X_{mi}^\tau, \\ \sum_{n,j,\tau'} \kappa_{\tau\tau'} D_{mi}^{\tau*} D_{jn}^{\tau'} X_{nj}^{\tau'} + (\epsilon_m^\tau - \epsilon_i^\tau) Y_{mi}^\tau + \sum_{n,j,\tau'} \kappa_{\tau\tau'} D_{mi}^{\tau*} D_{nj}^{\tau'} Y_{nj}^{\tau'} &= -\hbar\Omega Y_{mi}^\tau. \end{aligned} \quad (33)$$

The solution is

$$X_{mi}^\tau = \frac{D_{im}^{\tau*}}{\hbar\Omega - \epsilon_{mi}^\tau} K^\tau, \quad Y_{mi}^\tau = -\frac{D_{mi}^{\tau*}}{\hbar\Omega + \epsilon_{mi}^\tau} K^\tau \quad (34)$$

with $\epsilon_{mi}^\tau = \epsilon_m^\tau - \epsilon_i^\tau$ and $K^\tau = \sum_{\tau'} \kappa_{\tau\tau'} C^{\tau'}$.

The constant C^τ is defined as $C^\tau = \sum_{n,j} (D_{jn}^\tau X_{nj}^\tau + D_{nj}^\tau Y_{nj}^\tau)$. Using here the expressions for X_{nj}^τ and Y_{nj}^τ given above, one derives the useful relation

$$C^\tau = 2S^\tau K^\tau = 2S^\tau \sum_{\tau'} \kappa_{\tau\tau'} C^{\tau'}, \quad (35)$$

where the following notation is introduced:

$$S^\tau = \sum_{mi} |D_{mi}^\tau|^2 \frac{\epsilon_{mi}^\tau}{E^2 - (\epsilon_{mi}^\tau)^2} \quad (36)$$

with $E = \hbar\Omega$. Let us write out the relation (35) in detail

$$\begin{aligned} C^n - 2S^n(\kappa C^n + \bar{\kappa} C^p) &= 0, \\ C^p - 2S^p(\bar{\kappa} C^n + \kappa C^p) &= 0. \end{aligned} \quad (37)$$

The condition for existence of a nontrivial solution of this set of equations gives the secular equation

$$(1 - 2S^n \kappa)(1 - 2S^p \bar{\kappa}) - 4S^n S^p \bar{\kappa}^2 = 0. \quad (38)$$

Making obvious linear combinations of the two equations in (37), we write them in terms of isoscalar and isovector constants $C = C^n + C^p$, $\bar{C} = C^n - C^p$

$$\begin{aligned} C - 2(S^n + S^p)\kappa_0 C - 2(S^n - S^p)\kappa_1 \bar{C} &= 0, \\ \bar{C} - 2(S^n - S^p)\kappa_0 C - 2(S^n + S^p)\kappa_1 \bar{C} &= 0. \end{aligned} \quad (39)$$

Approximation (11) allows us to decouple the equations for isoscalar and isovector constants. Really, in this case $S^n = S^p \equiv S/2$; hence, we obtain two secular equations

$$1 - 2S\kappa_0 = 0, \quad \text{or} \quad 1 - S\kappa = S\bar{\kappa} \quad (40)$$

in the isoscalar case and

$$1 - 2S\kappa_1 = 0, \quad \text{or} \quad 1 - S\kappa = -S\bar{\kappa} \quad (41)$$

in the isovector one, the difference between them being in the strength constants only. Having in mind the relation $\kappa_1 = \alpha\kappa_0$, we come to the conclusion that it is sufficient to analyze the isovector case only – the results for isoscalar one are obtained by assuming $\alpha = 1$.

3.1 Eigenfrequencies

The detailed expression for the isovector secular equation is

$$\frac{1}{2\kappa_1} = \sum_{mi} |D_{mi}|^2 \frac{\epsilon_{mi}}{E^2 - \epsilon_{mi}^2}. \quad (42)$$

The operator D has only two types of nonzero matrix elements D_{mi} in the deformed oscillator basis. Matrix elements of the first type couple states of the same major shell. All corresponding transition energies are degenerate: $\epsilon_m - \epsilon_i = \hbar(\omega_x - \omega_z) \equiv \epsilon_0$. Matrix elements of the second type couple states of the different major shells with $\Delta N = 2$. All corresponding transition energies are degenerate too: $\epsilon_m - \epsilon_i = \hbar(\omega_x + \omega_z) \equiv \epsilon_2$. Therefore, the secular equation can be rewritten as

$$\frac{1}{2\kappa_1} = \frac{\epsilon_0 D_0}{E^2 - \epsilon_0^2} + \frac{\epsilon_2 D_2}{E^2 - \epsilon_2^2}. \quad (43)$$

The sums $D_0 = \sum_{mi(\Delta N=0)} |D_{mi}|^2$ and $D_2 = \sum_{mi(\Delta N=2)} |D_{mi}|^2$ can be calculated analytically (see Appendix B):

$$D_0 = \frac{Q_{00}}{m\bar{\omega}^2} \epsilon_0, \quad D_2 = \frac{Q_{00}}{m\bar{\omega}^2} \epsilon_2. \quad (44)$$

Let us transform the secular equation (43) in polynomial form

$$E^4 - E^2[(\epsilon_0^2 + \epsilon_2^2) + 2\kappa_1(\epsilon_0 D_0 + \epsilon_2 D_2)] + [\epsilon_0^2 \epsilon_2^2 + 2\kappa_1 \epsilon_0 \epsilon_2 (\epsilon_0 D_2 + \epsilon_2 D_0)] = 0.$$

Using here the expressions (44) for D_0 , D_2 and the self-consistent value of the strength constant (A.3), we find

$$E^4 - E^2(1 - \alpha/2)(\epsilon_0^2 + \epsilon_2^2) + (1 - \alpha)\epsilon_0^2\epsilon_2^2 = 0,$$

or

$$\Omega^4 - \Omega^2(2 - \alpha)\omega_+^2 + (1 - \alpha)\omega_-^4 = 0, \quad (45)$$

with the notation $\omega_+^2 = \omega_x^2 + \omega_z^2$ and $\omega_-^4 = (\omega_x^2 - \omega_z^2)^2$. This result coincides with that of [2]. By a trivial rearrangement of the terms in (45) one obtains the useful relation

$$\Omega^2(\Omega^2 - \omega_+^2) = (1 - \alpha)(\Omega^2\omega_+^2 - \omega_-^4). \quad (46)$$

Inserting expressions (A.3) for ω_x^2 , ω_z^2 into (45), we find $\omega_+^2 = 2\bar{\omega}^2(1 + \delta/3)$, $\omega_-^4 = 4\delta^2\bar{\omega}^4$ and reproduce formula (18) for the isovector case

$$\Omega^4 - 2\Omega^2\bar{\omega}^2(2 - \alpha)(1 + \delta/3) + 4\bar{\omega}^4(1 - \alpha)\delta^2 = 0.$$

Taking here $\alpha = 1$ we reproduce formula (14) for the isoscalar case

$$\Omega^4 - 2\Omega^2\bar{\omega}^2(1 + \delta/3) = 0.$$

3.2 B(E2)-factors

According to [3], the transition probability for the one-body operator $\hat{F} = \sum_{s=1}^A \hat{f}_s$ is calculated by means of the formulae

$$\langle 0 | \hat{F}^\tau | \nu \rangle = \sum_{mi} (f_{im}^\tau X_{mi}^{\tau,\nu} + f_{mi}^\tau Y_{mi}^{\tau,\nu}), \quad \langle \nu | \hat{F}^\tau | 0 \rangle = \sum_{mi} (f_{mi}^\tau X_{mi}^{\tau,\nu} + f_{im}^\tau Y_{mi}^{\tau,\nu}). \quad (47)$$

Quadrupole excitations are described by the operator (27) with $\hat{f}_{2\mu} = er^2 Y_{2\mu} = \tilde{e}D$, where $\tilde{e} = e\sqrt{\frac{5}{16\pi}}$. The expressions for X_{mi}^τ , Y_{mi}^τ are given by formulae (34). Combining these results we get

$$\langle 0 | \hat{F}_{21}^p | \nu \rangle = 2\tilde{e}K_\nu^p \sum_{mi} |D_{mi}^p|^2 \frac{\epsilon_{mi}^p}{E_\nu^2 - (\epsilon_{mi}^p)^2} = 2\tilde{e}K_\nu^p S_\nu^p = \tilde{e}C_\nu^p. \quad (48)$$

The constant C_ν^p is determined by the normalization condition

$$\delta_{\nu,\nu'} = \sum_{mi,\tau} (X_{mi}^{\tau,\nu*} X_{mi}^{\tau,\nu'} - Y_{mi}^{\tau,\nu*} Y_{mi}^{\tau,\nu'}),$$

that gives

$$\frac{1}{(C^p_\nu)^2} = E_\nu \sum_{mi} \left[\frac{|D^p_{mi}|^2}{(S^p_\nu)^2} \frac{\epsilon^p_{mi}}{[E^2_\nu - (\epsilon^p_{mi})^2]^2} + \frac{(C^n_\nu)^2}{(C^p_\nu)^2} \frac{|D^n_{mi}|^2}{(S^n_\nu)^2} \frac{\epsilon^n_{mi}}{[E^2_\nu - (\epsilon^n_{mi})^2]^2} \right]. \quad (49)$$

The ratio C^n/C^p is determined by any of the equations (37):

$$\frac{C^n}{C^p} = \frac{1 - 2S^p\kappa}{2S^p\bar{\kappa}} = \frac{2S^n\bar{\kappa}}{1 - 2S^n\kappa}. \quad (50)$$

Formula (49) is considerably simplified by the approximation (11), when $S^p = S^n \equiv S/2$, $\epsilon^p_{mi} = \epsilon^n_{mi}$, $D^p_{mi} = D^n_{mi}$. Applying the second forms of formulae (40, 41) it is easy to find that in this case $C^n/C^p = \pm 1$. As a result, the final expression for $B(E2)$ value is

$$B(E2)_\nu = 2 | \langle 0 | \hat{F}^p_{21} | \nu \rangle |^2 = 2\tilde{e}^2 \left(16E_\nu \kappa_1^2 \sum_{mi} |D_{mi}|^2 \frac{\epsilon_{mi}}{(E^2_\nu - \epsilon_{mi}^2)^2} \right)^{-1}. \quad (51)$$

With the help of formulae (44) this expression can be transformed into

$$\begin{aligned} B(E2)_\nu &= \frac{5}{8\pi} \frac{e^2 Q_{00}}{m\bar{\omega}^2 \alpha^2 E_\nu} \left[\frac{\epsilon_0^2}{(E_\nu^2 - \epsilon_0^2)^2} + \frac{\epsilon_2^2}{(E_\nu^2 - \epsilon_2^2)^2} \right]^{-1} \\ &= \frac{5}{8\pi} \frac{e^2 Q_{00}}{m\bar{\omega}^2 \alpha^2 E_\nu} \frac{(E_\nu^2 - \epsilon_0^2)^2 (E_\nu^2 - \epsilon_2^2)^2}{(E_\nu^2 - \epsilon_2^2)^2 \epsilon_0^2 + (E_\nu^2 - \epsilon_0^2)^2 \epsilon_2^2} \\ &= \frac{5}{16\pi} \frac{e^2 \hbar Q_{00}}{m\bar{\omega}^2 \Omega_\nu} \frac{(\Omega_\nu^2 \omega_+^2 - \omega_-^4)^2}{\Omega_\nu^4 \omega_+^2 - 2\Omega_\nu^2 \omega_-^4 + \omega_+^2 \omega_-^4}. \end{aligned} \quad (52)$$

At first sight, this expression has nothing in common with (31). Nevertheless, it can be shown that they are identical. To this end, we analyze carefully the denominator of the last expression in (52). Summing it with the secular equation (45) (multiplied by ω_+^2), which obviously does not change its value, we find after elementary combinations

$$\begin{aligned} \text{Denom} &= \Omega_\nu^4 \omega_+^2 - 2\Omega_\nu^2 \omega_-^4 + \omega_+^2 \omega_-^4 + \omega_+^2 [\Omega_\nu^4 - \Omega_\nu^2 (2 - \alpha) \omega_+^2 + (1 - \alpha) \omega_-^4] \\ &= \omega_+^2 \Omega_\nu^2 [2\Omega_\nu^2 - (2 - \alpha) \omega_+^2] - \omega_-^4 [2\Omega_\nu^2 - (2 - \alpha) \omega_+^2] \\ &= (\Omega_\nu^2 \omega_+^2 - \omega_-^4) [2\Omega_\nu^2 - (2 - \alpha) \omega_+^2]. \end{aligned} \quad (53)$$

This result allows us to write the final expression

$$B(E2)_\nu = \frac{5}{16\pi} \frac{e^2 \hbar}{m\bar{\omega}^2} Q_{00} \frac{\Omega_\nu^2 \omega_+^2 - \omega_-^4}{\Omega_\nu [2\Omega_\nu^2 - (2 - \alpha) \omega_+^2]}, \quad (54)$$

which coincides with (31) (we recall that $\omega_+^2 = 2\bar{\omega}^2(1 + \delta/3)$, $\omega_-^4 = 4\delta^2 \bar{\omega}^4$). By the simple transformations this formula is reduced to the result of Hamamoto and Nazarewicz [2] (taking into account, that they published it without the constant factor $\frac{5}{32\pi} \frac{e^2 \hbar}{m\omega_0} Q_{00}^0$).

3.3 B(M1)-factors

In accordance with formulae (22), (47), (34) the magnetic transition matrix element is

$$\langle 0|\hat{F}_{11}^p|\nu\rangle = K_\nu^p \sum_{mi} \left[\frac{(\hat{f}_{11}^p)_{im} D_{im}^{p*}}{E_\nu - \epsilon_{mi}^p} - \frac{(\hat{f}_{11}^p)_{mi} D_{mi}^{p*}}{E_\nu + \epsilon_{mi}^p} \right]. \quad (55)$$

As it is shown in Appendix B, the matrix element $(\hat{f}_{11}^p)_{im}$ is proportional to D_{im}^p (formula (B.16)). So, expression (55) is reduced to

$$\begin{aligned} \langle 0|\hat{F}_{11}^p|\nu\rangle &= -K_\nu^p \frac{\tilde{e}\hbar}{2c\sqrt{5}} (\omega_x^2 - \omega_z^2)^p \sum_{mi} \left[\frac{D_{im}^p D_{im}^{p*}}{\epsilon_{im}^p (E_\nu - \epsilon_{mi}^p)} - \frac{D_{mi}^p D_{mi}^{p*}}{\epsilon_{mi}^p (E_\nu + \epsilon_{mi}^p)} \right] \\ &= K_\nu^p \frac{\tilde{e}\hbar}{c\sqrt{5}} (\omega_x^2 - \omega_z^2)^p E_\nu \sum_{mi} \frac{|D_{mi}^p|^2}{\epsilon_{mi}^p [E_\nu^2 - (\epsilon_{mi}^p)^2]}. \end{aligned} \quad (56)$$

With the help of approximation (11) and the expressions (44) for D_0 , D_2 we find

$$\begin{aligned} \langle 0|\hat{F}_{11}^p|\nu\rangle &= \frac{C_\nu^p}{2S_\nu^p} \frac{\tilde{e}\hbar}{c\sqrt{5}} (\omega_x^2 - \omega_z^2) \frac{Q_{00}}{2m\bar{\omega}^2} \left(\frac{E_\nu}{E_\nu^2 - \epsilon_0^2} + \frac{E_\nu}{E_\nu^2 - \epsilon_2^2} \right) \\ &= -2\kappa_1 C_\nu^p \frac{\tilde{e}}{c\sqrt{5}} (\omega_x^2 - \omega_z^2) \frac{Q_{00}}{m\bar{\omega}^2} \frac{\Omega_\nu (\Omega_\nu^2 - \omega_+^2)}{\alpha (\Omega_\nu^2 \omega_+^2 - \omega_-^4)} \\ &= \frac{C_\nu^p}{2} \frac{\tilde{e}}{c\sqrt{5}} (\omega_x^2 - \omega_z^2) \frac{1 - \alpha}{\Omega_\nu}. \end{aligned} \quad (57)$$

Relation (46) and the self-consistent value of the strength constant $\kappa_1 = \alpha\kappa_0$ were used in the last step. For the magnetic transition probability we have

$$B(M1)_\nu = 2 |\langle 0|\hat{F}_{11}^p|\nu\rangle|^2 = 2 \frac{(C_\nu^p)^2}{4} \frac{\tilde{e}^2}{5c^2} \omega_-^4 \frac{(1 - \alpha)^2}{\Omega_\nu^2} = \frac{\omega_-^4}{20c^2} \frac{(1 - \alpha)^2}{\Omega_\nu^2} B(E2)_\nu. \quad (58)$$

This relation between $B(M1)$ and $B(E2)$ was also found (up to the factor $1/(20c^2)$) by Hamamoto and Nazarewicz [2]. Substituting expression (54) for $B(E2)$ into (58) we reproduce (with the help of relation (46)) formula (26).

3.4 “Synthetic” scissors and spurious state

The nature of collective excitations calculated with the method of Wigner function moments is quite easily revealed analyzing the roles of collective variables describing the phenomenon. The solution of this problem in the RPA approach is not so obvious. That is why the nature of the low-lying states has often been established by considering overlaps of these states with the “pure scissors state” [11, 12] or “synthetic state” [2] produced by the action of the scissors operator

$$\hat{S}_x = \mathcal{N}^{-1} (\langle I_x^{n2} \rangle \hat{I}_x^p - \langle I_x^{p2} \rangle \hat{I}_x^n)$$

on the ground state

$$|Syn\rangle = \hat{S}_x|0\rangle.$$

In the considered model the overlap of the “synthetic” state with the real scissors mode (and with IVGQR) can be calculated analytically. Surprisingly, it was not done until now. Let us at first modify the definition of the “synthetic” state. Due to axial symmetry one can use the \hat{I}_y^τ component instead of \hat{I}_x^τ , or any of their linear combinations, for example, the $\mu = 1$ component of the magnetic operator $\hat{F}_{1\mu}^\tau$, which is much more convenient for us. The terms $\langle I_x^{\tau 2} \rangle$ are introduced to ensure the orthogonality of the synthetic scissors to the spurious state $|Sp\rangle = (\hat{I}^n + \hat{I}^p)|0\rangle$. However, we do not need these terms because the collective states $|\nu\rangle$ of our model are already orthogonal to $|Sp\rangle$ (see below); hence, the overlaps $\langle Syn|\nu\rangle$ will be free from any admixtures of $|Sp\rangle$. So, we use the following definitions of the synthetic and spurious states:

$$|Syn\rangle = \mathcal{N}^{-1}(\hat{F}_{11}^p - \hat{F}_{11}^n)|0\rangle, \quad |Sp\rangle = (\hat{F}_{11}^p + \hat{F}_{11}^n)|0\rangle.$$

Let us demonstrate the orthogonality of the spurious state to all the rest of the states $|\nu\rangle$. As the first step it is necessary to show that the secular equation (38) has the solution $E = 0$. We need the expression for $S^\tau(E = 0) \equiv S^\tau(0)$. In accordance with (36), we have

$$S^\tau(E) = \left[\frac{\epsilon_0 D_0}{E^2 - \epsilon_0^2} + \frac{\epsilon_2 D_2}{E^2 - \epsilon_2^2} \right]^\tau, \quad S^\tau(0) = - \left[\frac{D_0}{\epsilon_0} + \frac{D_2}{\epsilon_2} \right]^\tau.$$

The expressions for D_0^τ , D_2^τ are easily extracted from formulae (B.10), (B.11):

$$D_0^\tau = \frac{\hbar}{m} Q_{00}^\tau \left[\frac{1 + \frac{4}{3}\delta}{\omega_x} - \frac{1 - \frac{2}{3}\delta}{\omega_z} \right]^\tau, \quad D_2^\tau = \frac{\hbar}{m} Q_{00}^\tau \left[\frac{1 + \frac{4}{3}\delta}{\omega_x} + \frac{1 - \frac{2}{3}\delta}{\omega_z} \right]^\tau. \quad (59)$$

So we find

$$\begin{aligned} S^\tau(0) &= -\frac{\hbar}{m} Q_{00}^\tau \left[\frac{1 + \frac{4}{3}\delta}{\omega_x} \left(\frac{1}{\epsilon_2} + \frac{1}{\epsilon_0} \right) + \frac{1 - \frac{2}{3}\delta}{\omega_z} \left(\frac{1}{\epsilon_2} - \frac{1}{\epsilon_0} \right) \right]^\tau \\ &= -\frac{\hbar^2}{m} \frac{4\delta^\tau Q_{00}^\tau}{\epsilon_2^\tau \epsilon_0^\tau} = -\frac{1}{m} \frac{3Q_{20}^\tau}{(\omega_x^2 - \omega_z^2)^\tau}, \end{aligned} \quad (60)$$

where, in accordance with (B.12),

$$(\omega_x^2 - \omega_z^2)^p = -\frac{6}{m}(\kappa Q_{20}^p + \bar{\kappa} Q_{20}^n), \quad (\omega_x^2 - \omega_z^2)^n = -\frac{6}{m}(\kappa Q_{20}^n + \bar{\kappa} Q_{20}^p). \quad (61)$$

Finally, we get

$$2S^p(0) = \frac{Q_{20}^p}{\kappa Q_{20}^p + \bar{\kappa} Q_{20}^n}, \quad 1 - 2S^p(0)\kappa = \frac{\bar{\kappa} Q_{20}^n}{\kappa Q_{20}^p + \bar{\kappa} Q_{20}^n},$$

$$2S^n(0) = \frac{Q_{20}^n}{\kappa Q_{20}^n + \bar{\kappa} Q_{20}^p}, \quad 1 - 2S^n(0)\kappa = \frac{\bar{\kappa} Q_{20}^p}{\kappa Q_{20}^n + \bar{\kappa} Q_{20}^p}.$$

It is easy to see that substituting these expressions into (38) we obtain an identity; therefore, the secular equation has a zero energy solution.

For the second step it is necessary to calculate the overlap $\langle Sp|\nu \rangle$. Summing (56) with an analogous expression for neutrons, we get

$$\begin{aligned} \langle Sp|\nu \rangle &= \frac{\tilde{\hbar}}{c\sqrt{5}} E_\nu \sum_\tau K_\nu^\tau (\omega_x^2 - \omega_z^2)^\tau \sum_{mi} \frac{|D_{mi}^\tau|^2}{\epsilon_{mi}^\tau (E_\nu^2 - \epsilon_{mi}^2)^\tau} \\ &= \frac{\tilde{\hbar}}{c\sqrt{5}} E_\nu \sum_\tau K_\nu^\tau (\omega_x^2 - \omega_z^2)^\tau \sum_{mi} \frac{|D_{mi}^\tau|^2 \epsilon_{mi}^\tau}{(\epsilon_{mi}^2)^\tau (E_\nu^2 - \epsilon_{mi}^2)^\tau}. \end{aligned} \quad (62)$$

Applying the algebraical identity

$$\frac{1}{\epsilon^2(E^2 - \epsilon^2)} = \frac{1}{E^2} \left(\frac{1}{\epsilon^2} + \frac{1}{E^2 - \epsilon^2} \right)$$

and remembering the definition (36) of S^τ we rewrite (62) as

$$\begin{aligned} \langle Sp|\nu \rangle &= \frac{\tilde{\hbar}}{c\sqrt{5} E_\nu} \sum_\tau K_\nu^\tau (\omega_x^2 - \omega_z^2)^\tau (S^\tau - S^\tau(0)) \\ &= \frac{\tilde{\hbar}}{c\sqrt{5}} \frac{K_\nu^p}{E_\nu} \left[(\omega_x^2 - \omega_z^2)^p (S^p - S^p(0)) + (\omega_x^2 - \omega_z^2)^n (S^n - S^n(0)) \frac{K_\nu^n}{K_\nu^p} \right]. \end{aligned} \quad (63)$$

In accordance with (35) and (50),

$$\frac{K_\nu^n}{K_\nu^p} = \frac{1 - 2S^p\kappa}{2S^n\bar{\kappa}}. \quad (64)$$

Noting now (see formula (60)) that $(\omega_x^2 - \omega_z^2)^\tau S^\tau(0) = -\frac{3}{m} Q_{20}^\tau$ and taking into account relations (61), we find

$$\begin{aligned} \langle Sp|\nu \rangle &= \beta \left\{ [(\kappa Q_2^p + \bar{\kappa} Q_2^n) 2S^p - Q_2^p] + [(\kappa Q_2^n + \bar{\kappa} Q_2^p) 2S^n - Q_2^n] \frac{1 - 2S^p\kappa}{2S^n\bar{\kappa}} \right\} \\ &= \beta \left\{ [(2S^p\kappa - 1) Q_2^p + 2S^p\bar{\kappa} Q_2^n] + [(2S^n\kappa - 1) Q_2^n + 2S^n\bar{\kappa} Q_2^p] \frac{1 - 2S^p\kappa}{2S^n\bar{\kappa}} \right\} \\ &= \beta \left\{ 2S^p\bar{\kappa} Q_2^n + (2S^n\kappa - 1) Q_2^n \frac{1 - 2S^p\kappa}{2S^n\bar{\kappa}} \right\} \\ &= \beta \frac{Q_2^n}{2S^n\bar{\kappa}} \{ 2S^n\bar{\kappa} 2S^p\bar{\kappa} - (1 - 2S^n\kappa)(1 - 2S^p\kappa) \} = 0, \end{aligned} \quad (65)$$

where $\beta = -\frac{3}{m} \frac{\tilde{\hbar}}{c\sqrt{5}} \frac{K_\nu^p}{E_\nu}$ and $Q_2 \equiv Q_{20}$. The expression in the last curly brackets coincides obviously with the secular equation (38) that proves the orthogonality of the spurious state to all physical states of the considered model. So we can conclude that strictly speaking this is not a spurious state, but one of the exact eigenstates of the model corresponding to

the integral of motion $I^n + I^p$. In other words [3]: "In fact these excitations are not really spurious, but they represent a different type of motion which has to be treated separately." The same conclusion was made by N. Lo Iudice [13] who solved this problem approximately with the help of several assumptions (a small deformation limit, for example).

The problem of the "spurious" state being solved, the calculation of the overlaps $\langle Syn|\nu \rangle$ becomes trivial. Really, we have shown that $\langle 0|\hat{F}_{11}^n + \hat{F}_{11}^p|\nu \rangle = 0$. That means that $\langle 0|\hat{F}_{11}^n|\nu \rangle = -\langle 0|\hat{F}_{11}^p|\nu \rangle$; hence, $\langle Syn|\nu \rangle = \mathcal{N}^{-1} \langle 0|\hat{F}_{11}^p - \hat{F}_{11}^n|\nu \rangle = 2\mathcal{N}^{-1} \langle 0|\hat{F}_{11}^p|\nu \rangle$ and

$$U^2 \equiv |\langle Syn|\nu \rangle|^2 = 2\mathcal{N}^{-2}B(M1)_\nu. \quad (66)$$

The nontrivial part of the problem is the calculation of the normalization factor \mathcal{N} . It is important not to forget about the time dependence of the synthetic state which should be determined by the external field:

$$|Syn(t)\rangle = \mathcal{N}^{-1}[(\hat{F}_{11}^p - \hat{F}_{11}^n)e^{-i\Omega t} + (\hat{F}_{11}^p - \hat{F}_{11}^n)^\dagger e^{i\Omega t}]|0\rangle.$$

Then we have

$$\begin{aligned} \mathcal{N}^2 &= 2 \langle 0|(\hat{F}_{11}^p - \hat{F}_{11}^n)^\dagger(\hat{F}_{11}^p - \hat{F}_{11}^n)|0\rangle \\ &= 2 \sum_{ph} \langle 0|(\hat{F}_{11}^p - \hat{F}_{11}^n)^\dagger|ph\rangle \langle ph|(\hat{F}_{11}^p - \hat{F}_{11}^n)|0\rangle = 2 \sum_{ph} |\langle ph|(\hat{F}_{11}^p - \hat{F}_{11}^n)|0\rangle|^2 \\ &= 2 \sum_{\tau, ph} |\langle ph|\hat{F}_{11}^\tau|0\rangle|^2 = 2 \sum_{\tau, ph} |(f_{11}^\tau)_{ph}|^2. \end{aligned} \quad (67)$$

With the help of relation (B.16) we find

$$\begin{aligned} \mathcal{N}^2 &= \frac{2}{5} \left(\frac{e\hbar}{2c} \right)^2 \sum_{\tau, ph} \left(\omega_-^4 \frac{|\langle ph|r^2 Y_{21}|0\rangle|^2}{\epsilon_{ph}^2} \right)^\tau \\ &= \frac{1}{8\pi} \left(\frac{e\hbar}{2c} \right)^2 \sum_{\tau} (\omega_-^4)^\tau \left(\frac{D_0}{\epsilon_0^2} + \frac{D_2}{\epsilon_2^2} \right)^\tau. \end{aligned} \quad (68)$$

Expressions for D_0^τ , D_2^τ , ω_x^τ , ω_z^τ are given by formulae (59), (B.12). To get a definite number, it is necessary to make some assumption concerning the relation between neutron and proton equilibrium characteristics. As usual, we apply the approximation (11), i.e., suppose $Q_{00}^n = Q_{00}^p$, $Q_{20}^n = Q_{20}^p$. It is easy to check that in this case formulae for $\omega_{x,z}^\tau$ are reduced to the ones for the isoscalar case, namely (A.3), and $D_0^\tau = D_0/2$, $D_2^\tau = D_2/2$, where D_0 and D_2 are given by (44). So we get

$$\mathcal{N}^2 = \frac{\omega_-^4}{8\pi} \left(\frac{e\hbar}{2c} \right)^2 \frac{Q_{00}}{m\bar{\omega}^2} \left(\frac{1}{\epsilon_0} + \frac{1}{\epsilon_2} \right) = \frac{\delta}{2\pi} \frac{m\omega_x}{\hbar} Q_{00} \mu_N^2. \quad (69)$$

The estimation of the overlap for ^{156}Gd with $\delta = 0.27$ gives $\mathcal{N}^2 = 34.72\mu_N^2$ and $U^2 = 0.53$ (see eq. (66)), that is two times larger than the result of [11] obtained in QRPA calculations with the Skyrme forces. The disagreement can naturally be attributed to the difference in forces and especially to the lack of pair correlations in our approach. In a small deformation limit $U^2 = \frac{1}{2}\sqrt{\frac{3}{2}} \approx 0.6$.

This is the maximum possible overlap of the "pure" (or "synthetic") scissors with the real scissors. The increasing of δ and /or taking into account pairing correlations decreases its value, that is confirmed by numerous microscopic calculations with various forces [17]. Such small overlap leads inevitably to the conclusion, that the original model of counter rotating rigid rotors [10] has not very much in common with the real scissors mode, the correct description of which requires the proper treatment of the Fermi surface deformation and the coupling with IVGQR.

3.4.1 Superdeformation

A certain drawback of our approach is that, so far, we have not included the superfluidity into our description. Nevertheless, our formulae (20, 26) can be successfully used for the description of superdeformed nuclei where the pairing is very weak [2, 10]. For example, applying them to the superdeformed nucleus ^{152}Dy ($\delta \simeq 0.6$, $\hbar\omega_0 = 41/A^{1/3}\text{MeV}$), we get

$$E_{iv} = 20.8 \text{ MeV}, \quad B(M1)_{iv} = 15.9 \mu_N^2$$

for the isovector GQR and

$$E_{sc} = 4.7 \text{ MeV}, \quad B(M1)_{sc} = 20.0 \mu_N^2$$

for the scissors mode. There are not so many results of other calculations to compare with. As a matter of fact, there are only two papers considering this problem.

The phenomenological TRM model [10] predicts

$$E_{iv} \simeq 26 \text{ MeV}, \quad B(M1)_{iv} \simeq 26 \mu_N^2, \quad E_{sc} \simeq 6.1 \text{ MeV}, \quad B(M1)_{sc} \simeq 22 \mu_N^2.$$

The only existing microscopic calculation [2] in the framework of QRPA with separable forces gives

$$E_{iv} \simeq 28 \text{ MeV}, \quad B(M1)_{iv} \simeq 37 \mu_N^2, \quad E_{sc} \simeq 5 - 6 \text{ MeV}, \quad B(M1)_{1+} \simeq 23 \mu_N^2.$$

Here $B(M1)_{1+}$ denotes the total $M1$ orbital strength carried by the calculated $K^\pi = 1^+$ QRPA excitations modes in the energy region below 20 MeV.

It is easy to see that in the case of IVGQR one can speak, at least, about qualitative agreement. Our results for E_{sc} and $B(M1)_{sc}$ are in good agreement with that of phenomenological model and with E_{sc} and $B(M1)_{1+}$ of Hamamoto and Nazarewicz.

It is possible to extract from the histogram of [2] the value of the overlap of calculated low-lying 1^+ excitations with the synthetic scissors state: $|\langle Syn|1^+ \rangle|^2 \approx 0.4$. The result of our calculation $U^2 = 0.43$ agrees with it very well. So the natural conclusion of this section is that the correct treatment of pair correlations is obligatory for a reasonable description of the scissors mode.

3.5 Equations of motion

Let us look on WFM equations of motion from the RPA point of view. Is it possible to construct something similar in the RPA approach? Equations (10) are written for average values of operators and are valid for the description of the arbitrary amplitude motion. One can compare with RPA only their linearised version, obtained by the variation of equations. The variables of linearised equations are the variations of the above mentioned average values. It is natural to suppose some correspondence between the variation of the average value of \hat{F} operator and the matrix element of the type $\langle 0|\hat{F}|\nu \rangle$ used to calculate transition probabilities. To check this idea we have to derive dynamical equations for matrix elements of the operators $r_{\lambda\mu}^2$, $\hat{p}_{\lambda\mu}^2$ and $(r\hat{p})_{\lambda\mu}$ to compare them with linearised equations (10). To this end we combine RPA equations (33) in accordance with the definition (47) of matrix elements:

$$\hbar\Omega_\nu \sum_{mi} (f_{im}^\tau X_{mi}^{\tau,\nu} + f_{mi}^\tau Y_{mi}^{\tau,\nu}) = \sum_{mi} \epsilon_{mi} (f_{im}^\tau X_{mi}^{\tau,\nu} - f_{mi}^\tau Y_{mi}^{\tau,\nu}) + K_\nu^\tau \sum_{mi} (f_{im}^\tau D_{im}^{\tau*} - f_{mi}^\tau D_{mi}^{\tau*}). \quad (70)$$

Taking into account the relations

$$\epsilon_{mi} f_{im} = [\hat{f}, H_0]_{im}, \quad \epsilon_{mi} f_{mi} = -[\hat{f}, H_0]_{mi},$$

one rewrites this equation as

$$\hbar\Omega_\nu \langle 0|\hat{F}^\tau|\nu \rangle = \sum_{mi} \{[\hat{f}^\tau, H_0^\tau]_{im} X_{mi}^{\tau,\nu} + [\hat{f}^\tau, H_0^\tau]_{mi} Y_{mi}^{\tau,\nu} + K_\nu^\tau (f_{im}^\tau D_{im}^{\tau*} - f_{mi}^\tau D_{mi}^{\tau*})\}. \quad (71)$$

The Hamiltonian of the axially deformed harmonic oscillator corresponding to the mean field (8) is

$$H_0^\tau(\mathbf{r}) = \sum_{s=1}^{N_\tau} \left\{ \frac{\hat{\mathbf{p}}_s^2}{2m} + \frac{1}{2} m \omega^2 \mathbf{r}_s^2 + Z_{20}^\tau(eq) r_{20}^2(s) \right\}. \quad (72)$$

Let us consider the operator $\hat{f} = \sqrt{6} r_{21}^2 = q_{21} = D$. Calculating the commutator

$$[r_{21}^2, H_0] = i\hbar \frac{2}{m} (r\hat{p})_{21}$$

we find from (71) the following equation

$$\begin{aligned} \hbar\Omega_\nu < 0 | \sum_{s=1}^{N_\tau} D_s^\tau | \nu > &= i\hbar\sqrt{6} \frac{2}{m} \sum_{mi} \{ ((r\hat{p})_{21})_{im}^\tau X_{mi}^{\tau,\nu} + ((r\hat{p})_{21})_{mi}^\tau Y_{mi}^{\tau,\nu} \} \\ &+ K_\nu^\tau \sum_{mi} (D_{im}^\tau D_{im}^{\tau*} - D_{mi}^\tau D_{mi}^{\tau*}). \end{aligned} \quad (73)$$

Taking into account relations $(D^*)_{im} = (D)_{mi}^*$ and $|D_{mi}|^2 = |D_{im}|^2$ we find, that the last sum in (73) is equal to zero. Applying again formula (47) and introducing the notation $\hat{R}_{\lambda\mu} = \sum_{s=1}^A (r_s^2)_{\lambda\mu}$, $\hat{L}_{\lambda\mu} = \sum_{s=1}^A (r_s \hat{p}_s)_{\lambda\mu}$ we write (73) as

$$-i\Omega_\nu < 0 | \hat{R}_{21}^\tau | \nu > = \frac{2}{m} < 0 | \hat{L}_{21}^\tau | \nu >. \quad (74)$$

Identifying the matrix elements $< 0 | \hat{R}_{21}^\tau | \nu >$ and $< 0 | \hat{L}_{21}^\tau | \nu >$ with \mathcal{R}_{21}^τ and \mathcal{L}_{21}^τ respectively we reproduce the variation of the first equation in (10) (having in mind the time dependence via $e^{-i\Omega t}$).

Let us consider the operator $\hat{f} = (r\hat{p})_{21}$. The required commutator is evaluated to be

$$[(r\hat{p})_{21}, H_0] = i\frac{\hbar}{m} \hat{p}_{21}^2 - i\hbar m \omega^2 r_{21}^2 - i\frac{\hbar}{\sqrt{6}} Z_{20}(eq) r_{21}^2.$$

With this result equation (71) looks as

$$\begin{aligned} \hbar\Omega_\nu < 0 | \hat{L}_{21}^\tau | \nu > &= i\frac{\hbar}{m} < 0 | \hat{P}_{21}^\tau | \nu > - i\hbar m \omega^2 < 0 | \hat{R}_{21}^\tau | \nu > \\ &- i\frac{\hbar}{\sqrt{6}} Z_{20}^\tau(eq) < 0 | \hat{R}_{21}^\tau | \nu > + K_\nu^\tau \sum_{mi} [((r\hat{p})_{21})_{im}^\tau D_{im}^{\tau*} - ((r\hat{p})_{21})_{mi}^\tau D_{mi}^{\tau*}], \end{aligned} \quad (75)$$

where the notation $\hat{P}_{\lambda\mu} = \sum_{s=1}^A (\hat{p}_s^2)_{\lambda\mu}$ has been introduced. The last sum is calculated with the help of formula (B.17). Using the fact, that $\epsilon_{im} = -\epsilon_{mi}$, one gets

$$\begin{aligned} \sum_{mi} [((r\hat{p})_{21})_{im}^\tau D_{im}^{\tau*} - ((r\hat{p})_{21})_{mi}^\tau D_{mi}^{\tau*}] &= -i\frac{m}{2\hbar} \sum_{mi} \epsilon_{mi} [(r_{21}^2)_{im}^\tau D_{im}^{\tau*} + (r_{21}^2)_{mi}^\tau D_{mi}^{\tau*}] \\ &= -i\frac{m}{\hbar\sqrt{6}} \sum_{mi} \epsilon_{mi}^\tau |D_{mi}^\tau|^2 = -i\frac{m}{\hbar\sqrt{6}} (\epsilon_0^\tau D_0^\tau + \epsilon_2^\tau D_2^\tau) = -i\hbar \frac{4}{\sqrt{6}} (1 + \delta/3) Q_{00}^\tau. \end{aligned}$$

According to the definitions (see formula 35) we have

$$K_\nu^n = \sum_\tau \kappa_{n\tau} C_\nu^\tau = (\chi < 0 | \hat{R}_{21}^n | \nu > + \bar{\chi} < 0 | \hat{R}_{21}^p | \nu >) / \sqrt{6},$$

$$K_\nu^p = \sum_\tau \kappa_{p\tau} C_\nu^\tau = (\chi < 0 | \hat{R}_{21}^p | \nu > + \bar{\chi} < 0 | \hat{R}_{21}^n | \nu >) / \sqrt{6}.$$

So, the equation (75) (let us say, for neutrons) is transformed into

$$\begin{aligned} -i\Omega_\nu < 0 | \hat{L}_{21}^n | \nu > &= +\frac{1}{m} < 0 | \hat{P}_{21}^n | \nu > - m\omega^2 < 0 | \hat{R}_{21}^n | \nu > \\ -\frac{1}{\sqrt{6}} Z_{20}^n(eq) < 0 | \hat{R}_{21}^n | \nu > &- \frac{2}{3} (1 + \delta/3) Q_{00}^n (\chi < 0 | \hat{R}_{21}^n | \nu > + \bar{\chi} < 0 | \hat{R}_{21}^p | \nu >). \end{aligned} \quad (76)$$

The equation for protons is obtained by interchanging indices n and p. One has to compare this equation with the variation of the second equation in (10) with $\lambda = 2$, $\mu = 1$. Let us write this variation in detail:

$$\frac{d}{dt} \mathcal{L}_{21}^\tau - \frac{1}{m} \mathcal{P}_{21}^\tau + m\omega^2 \mathcal{R}_{21}^\tau - 2\sqrt{5} \sum_{j=0,2} \sqrt{2j+1} \{_{221}^{11j}\} \sum_{\sigma,\nu} C_{2\sigma,j\nu}^{21} [Z_{2\sigma}^\tau(eq) \mathcal{R}_{j\nu}^\tau + \delta Z_{2\sigma}^\tau R_{j\nu}^\tau(eq)] = 0.$$

We recall, that only $R_{00}^\tau(eq)$ and $R_{20}^\tau(eq)$ have non zero values, so this equation is reduced to

$$\begin{aligned} \frac{d}{dt} \mathcal{L}_{21}^\tau - \frac{1}{m} \mathcal{P}_{21}^\tau + m\omega^2 \mathcal{R}_{21}^\tau - 10 \{_{221}^{112}\} C_{20,21}^{21} Z_{20}^\tau(eq) \mathcal{R}_{21}^\tau \\ - 2\sqrt{5} \delta Z_{21}^\tau [\{_{221}^{110}\} C_{21,00}^{21} R_{00}^\tau(eq) + \sqrt{5} \{_{221}^{112}\} C_{21,20}^{21} R_{20}^\tau(eq)] = 0. \end{aligned}$$

In agreement with definition (8) of $Z_{\lambda\mu}^\tau$ its variation is

$$\delta Z_{2\mu}^n = \chi \mathcal{R}_{2\mu}^n + \bar{\chi} \mathcal{R}_{2\mu}^p, \quad \delta Z_{2\mu}^p = \chi \mathcal{R}_{2\mu}^p + \bar{\chi} \mathcal{R}_{2\mu}^n.$$

Substituting 6j-symbols and Clebsch-Gordan coefficients by their numerical values we obtain finally (e.g. for neutrons)

$$\frac{d}{dt} \mathcal{L}_{21}^n - \frac{1}{m} \mathcal{P}_{21}^n + m\omega^2 \mathcal{R}_{21}^n + \frac{1}{\sqrt{6}} Z_{20}^n(eq) \mathcal{R}_{21}^n + \frac{2}{3} (1 + \delta/3) Q_{00}^n (\chi \mathcal{R}_{21}^n + \bar{\chi} \mathcal{R}_{21}^p) = 0.$$

This equation coincides obviously with (76) if to assume the time dependence via $e^{-i\Omega t}$ and to identify the matrix elements $< 0 | \hat{R}_{21}^\tau | \nu >$, $< 0 | \hat{L}_{21}^\tau | \nu >$ and $< 0 | \hat{P}_{21}^\tau | \nu >$ with the variables \mathcal{R}_{21}^τ , \mathcal{L}_{21}^τ and \mathcal{P}_{21}^τ , respectively.

Let us consider the operator $\hat{f} = (\hat{p}^2)_{21}$. The required commutator is

$$[(\hat{p}^2)_{21}, H_0] = -i\hbar 2m\omega^2 (r\hat{p})_{21} + i\hbar 4\sqrt{5} \sum_{j=1}^2 \sqrt{2j+1} \{_{221}^{11j}\} C_{20,j1}^{21} Z_{20}(eq) (r\hat{p})_{j1}$$

and one obtains from (71) the following equation

$$\begin{aligned}
\hbar\Omega_\nu < 0 | \hat{P}_{21}^\tau | \nu > &= -i\hbar 2m\omega^2 < 0 | \hat{L}_{21}^\tau | \nu > \\
&+ i\hbar 4\sqrt{5} \sum_{j=1}^2 \sqrt{2j+1} \{^{11j}_{221}\} C_{20,j1}^{21} Z_{20}^\tau(eq) < 0 | \hat{L}_{j1}^\tau | \nu > \\
&+ K_\nu^\tau \sum_{mi} [((\hat{p}^2)_{21}^\tau)_{im} D_{im}^{\tau*} - ((\hat{p}^2)_{21}^\tau)_{mi} D_{mi}^{\tau*}]. \tag{77}
\end{aligned}$$

It is easy to show (with the help of formula (B.18)) that the last sum is equal to zero. This equation must be compared with the variation of the last equation in (10) with $\lambda = 2, \mu = 1$. Let us write it in detail. Taking into account that $L_{\lambda\mu}^\tau(eq) = 0$ we find the equation

$$\frac{d}{dt} \mathcal{P}_{21}^\tau + 2m\omega^2 \mathcal{L}_{21}^\tau - 4\sqrt{5} \sum_{j=1}^2 \sqrt{2j+1} \{^{11j}_{221}\} C_{20,j1}^{21} Z_{20}^\tau(eq) \mathcal{L}_{j1}^\tau = 0$$

that obviously coincides with (77) if to assume the $e^{-i\Omega t}$ time dependence and to identify the proper RPA matrix elements with the respective WFM variables.

4 WFM versus RPA

The exact relation between RPA matrix elements and the respective WFM variables can be established with the help of the linear response theory. Let us first recall, following Appendix D of [3], the necessary definitions concerning the density and the density matrix.

The density operator is defined as

$$\hat{\rho}(\mathbf{r}) = \sum_{s=1}^A \delta(\mathbf{r} - \hat{\mathbf{r}}_s) = \sum_{pq} d_{pq}(\mathbf{r}) a_p^\dagger a_q, \tag{78}$$

where $d_{pq}(\mathbf{r}) = \langle p | \delta(\mathbf{r} - \hat{\mathbf{r}}) | q \rangle = \sum_{\sigma,\tau} \phi_p^*(\mathbf{r}\sigma\tau) \phi_q(\mathbf{r}\sigma\tau)$ and $\phi_q(\mathbf{r}\sigma\tau)$ are single-particle wave functions. Indices p, q include spin and isospin quantum numbers σ and τ .

The density of particles in the system depends on its state Ψ and is defined as the average value of a density operator over this state:

$$\begin{aligned}
\rho(\mathbf{r}) &= \langle \Psi | \hat{\rho}(\mathbf{r}) | \Psi \rangle = \sum_{pq} d_{pq}(\mathbf{r}) \rho_{qp} \\
&= A \sum_{\sigma,\tau,\dots,\sigma_A,\tau_A} \int d^3r_2 \dots d^3r_A |\Psi(\mathbf{r}\sigma\tau, \mathbf{r}_2\sigma_2\tau_2, \dots, \mathbf{r}_A\sigma_A\tau_A)|^2, \tag{79}
\end{aligned}$$

where $\rho_{qp} = \langle \Psi | a_p^\dagger a_q | \Psi \rangle$. The particle density (79) can be interpreted as the diagonal element (in the coordinate space representation) of **the density matrix** which is defined as

$$\rho(\mathbf{r}\sigma\tau, \mathbf{r}'\sigma'\tau') = \sum_{pq} \phi_p^*(\mathbf{r}'\sigma'\tau') \phi_q(\mathbf{r}\sigma\tau) \langle \Psi | a_p^\dagger a_q | \Psi \rangle = \sum_{pq} d_{pq}(\mathbf{r}'\sigma'\tau', \mathbf{r}\sigma\tau) \rho_{qp} \quad (80)$$

with $d_{pq}(\mathbf{r}'\sigma'\tau', \mathbf{r}\sigma\tau) = \phi_p^*(\mathbf{r}'\sigma'\tau') \phi_q(\mathbf{r}\sigma\tau)$. The average value of the arbitrary one-body operator

$$\hat{F} = \sum_{s=1}^A \hat{f}_s = \sum_{pq} f_{pq} a_p^\dagger a_q \quad (81)$$

is written in terms of the density matrix as

$$\langle \Psi | \hat{F} | \Psi \rangle = \sum_{pq} f_{pq} \langle \Psi | a_p^\dagger a_q | \Psi \rangle = \sum_{pq} f_{pq} \rho_{qp} = \text{Tr}(f\rho).$$

Let us consider the system to be in the weak external time-dependent field

$$\hat{W}(t) = \hat{W} \exp(-i\Omega t) + \hat{W}^\dagger \exp(i\Omega t), \quad (82)$$

where $\hat{W} = \sum_{pq} w_{pq} a_p^\dagger a_q$ is a one-body operator. The change of the ground state wave function produced by this field is found by using the time-dependent perturbation theory [14]:

$$\Psi(t) = |0\rangle + \sum_{\nu} |\nu\rangle \left[c_{\nu} e^{-i\Omega t} - \bar{c}_{\nu}^* e^{i\Omega t} \right]. \quad (83)$$

Here $|0\rangle$ and $|\nu\rangle$ are stationary eigenstates of the unperturbed system and

$$c_{\nu} = \frac{\langle \nu | \hat{W} | 0 \rangle}{\hbar(\Omega - \Omega_{\nu})} = \sum_{pq} \frac{\langle \nu | a_p^\dagger a_q | 0 \rangle}{\hbar(\Omega - \Omega_{\nu})} w_{pq}, \quad \bar{c}_{\nu} = \frac{\langle 0 | \hat{W} | \nu \rangle}{\hbar(\Omega + \Omega_{\nu})} = \sum_{pq} \frac{\langle 0 | a_p^\dagger a_q | \nu \rangle}{\hbar(\Omega + \Omega_{\nu})} w_{pq}. \quad (84)$$

Inserting this expression into formula (80) we obtain the perturbed density matrix

$$\rho(\mathbf{r}\sigma\tau, \mathbf{r}'\sigma'\tau', t) = \rho_0(\mathbf{r}\sigma\tau, \mathbf{r}'\sigma'\tau') + \delta\rho(\mathbf{r}\sigma\tau, \mathbf{r}'\sigma'\tau', t),$$

where $\rho_0(\mathbf{r}\sigma\tau, \mathbf{r}'\sigma'\tau')$ is the unperturbed (equilibrium) density matrix

$$\rho_0(\mathbf{r}\sigma\tau, \mathbf{r}'\sigma'\tau') = \sum_{pq} d_{qp}(\mathbf{r}'\sigma'\tau', \mathbf{r}\sigma\tau) \langle 0 | a_q^\dagger a_p | 0 \rangle = \sum_{pq} d_{qp}(\mathbf{r}'\sigma'\tau', \mathbf{r}\sigma\tau) \rho_{pq}^{(0)}$$

and $\delta\rho(\mathbf{r}\sigma\tau, \mathbf{r}'\sigma'\tau', t)$ is the change of the density matrix

$$\delta\rho(\mathbf{r}\sigma\tau, \mathbf{r}'\sigma'\tau', t) = \sum_{pq} d_{qp}(\mathbf{r}'\sigma'\tau', \mathbf{r}\sigma\tau) \rho_{pq}^{(1)}(t) \quad (85)$$

with

$$\begin{aligned}\rho_{pq}^{(1)}(t) = & \sum_{\nu} \left[(\langle 0|a_q^\dagger a_p|\nu \rangle c_\nu - \langle \nu|a_q^\dagger a_p|0 \rangle \bar{c}_\nu) e^{-i\Omega t} \right. \\ & \left. + (\langle \nu|a_q^\dagger a_p|0 \rangle c_\nu^* - \langle 0|a_q^\dagger a_p|\nu \rangle \bar{c}_\nu^*) e^{i\Omega t} \right].\end{aligned}\quad (86)$$

Deriving (85) we neglected the terms proportional to $|\hat{W}|^2$. At this stage it is necessary to remind that we work in a Hartree-Fock approximation. That means that stationary states $|0 \rangle, |\nu \rangle$ are Slater determinants; matrix $\rho_{pq}^{(0)} = \rho_q \delta_{pq}$ is diagonal with $\rho_q = 1$ for levels below the Fermi level and $\rho_q = 0$ for levels above the Fermi level. The requirement $(\rho_0 + \delta\rho)^2 = (\rho_0 + \delta\rho)$ leads to the well known [3] property of the matrix $\rho_{pq}^{(1)}$: it has only particle-hole nonvanishing matrix elements. Looking to formula (86) we see that it is possible for the matrix elements $\langle 0|a_q^\dagger a_p|\nu \rangle$ to be different from zero only for particle-hole combinations of indices q, p . Consequently, the summation over p, q in formula (84) for c_n and \bar{c}_n will also be restricted only to particle-hole pairs. So we can write $\rho_{pq}^{(1)}$ as

$$\rho_{pq}^{(1)}(t) = \sum_{p'q'} \left[R_{pq,p'q'}(\Omega) e^{-i\Omega t} + R_{qp,p'q'}^*(\Omega) e^{i\Omega t} \right] w_{p'q'},$$

where

$$R_{pq,p'q'}(\Omega) = \sum_{\nu} \left(\frac{\langle 0|a_q^\dagger a_p|\nu \rangle \langle \nu|a_{p'}^\dagger a_{q'}|0 \rangle}{\hbar(\Omega - \Omega_\nu)} - \frac{\langle 0|a_{p'}^\dagger a_{q'}|\nu \rangle \langle \nu|a_q^\dagger a_p|0 \rangle}{\hbar(\Omega + \Omega_\nu)} \right)$$

is the RPA response function [3], where the index pairs pq and $p'q'$ are restricted to particle-hole pairs. For the change of the arbitrary operator average value we have:

$$\delta \langle \Psi | \hat{F} | \Psi \rangle = \sum_{pq} f_{pq} \rho_{qp}^{(1)}. \quad (87)$$

Now we are ready to analyze the WFM variables. The first one is

$$R_{\lambda\mu}^\tau(t) = 2(2\pi\hbar)^{-3} \int d^3p \int d^3r r_{\lambda\mu}^2 f^\tau(\mathbf{r}, \mathbf{p}, t).$$

Using here the definition (2) of the Wigner function and the definition of the δ -function we find

$$\begin{aligned}R_{\lambda\mu}^\tau(t) &= \frac{2}{(2\pi\hbar)^3} \int d^3r r_{\lambda\mu}^2 \int d^3s \int d^3p \exp(-i\mathbf{p} \cdot \mathbf{s}/\hbar) \rho^\tau(\mathbf{r} + \frac{\mathbf{s}}{2}, \mathbf{r} - \frac{\mathbf{s}}{2}, t) \\ &= 2 \int d^3r r_{\lambda\mu}^2 \rho^\tau(\mathbf{r}, \mathbf{r}, t) = \sum_{\sigma} \int d^3r r_{\lambda\mu}^2 \rho(\mathbf{r}\sigma\tau, \mathbf{r}\sigma\tau, t) \\ &= \sum_{pq} \sum_{\sigma} \int d^3r r_{\lambda\mu}^2 \phi_p^*(\mathbf{r}\sigma\tau) \phi_q(\mathbf{r}\sigma\tau) \langle \Psi | a_p^\dagger a_q | \Psi \rangle\end{aligned}$$

$$\begin{aligned}
&= \sum_{pq} (r_{\lambda\mu}^2)_{pq}^\tau \rho_{qp}(t) = \langle \Psi | \sum_{pq} (r_{\lambda\mu}^2)_{pq}^\tau a_p^\dagger a_q | \Psi \rangle \\
&= \langle \Psi | \sum_{s=1}^{N_\tau} r_{\lambda\mu}^2(s) | \Psi \rangle = \langle \Psi | \hat{R}_{\lambda\mu}^\tau | \Psi \rangle. \tag{88}
\end{aligned}$$

For the variation of this variable one can write the following chain of relations

$$\begin{aligned}
\delta R_{\lambda\mu}^\tau(t) &= \mathcal{R}_{\lambda\mu}^\tau(t) = 2 \int d^3r r_{\lambda\mu}^2 \delta \rho^\tau(\mathbf{r}, \mathbf{r}, t) \\
&= \sum_{pq} \sum_{\sigma} \int d^3r r_{\lambda\mu}^2 \phi_p^*(\mathbf{r}\sigma\tau) \phi_q(\mathbf{r}\sigma\tau) \rho_{qp}^{(1)}(t) = \sum_{pq} (r_{\lambda\mu}^2)_{pq}^\tau \rho_{qp}^{(1)}(t) \\
&= \sum_{\nu} (\langle 0 | \hat{R}_{\lambda\mu}^\tau | \nu \rangle c_{\nu} - \langle \nu | \hat{R}_{\lambda\mu}^\tau | 0 \rangle \bar{c}_{\nu}) e^{-i\Omega t} \\
&\quad + \sum_{\nu} (\langle \nu | \hat{R}_{\lambda\mu}^\tau | 0 \rangle c_{\nu}^* - \langle 0 | \hat{R}_{\lambda\mu}^\tau | \nu \rangle \bar{c}_{\nu}^*) e^{i\Omega t}. \tag{89}
\end{aligned}$$

For the second variable we have

$$\begin{aligned}
L_{\lambda\mu}^\tau(t) &= 2(2\pi\hbar)^{-3} \int d^3p \int d^3r (rp)_{\lambda\mu} f^\tau(\mathbf{r}, \mathbf{p}, t) \\
&= \frac{2}{(2\pi\hbar)^3} \int d^3r \int d^3s \int d^3p (rp)_{\lambda\mu} \exp(-i\mathbf{p} \cdot \mathbf{s}/\hbar) \rho^\tau(\mathbf{r} + \frac{\mathbf{s}}{2}, \mathbf{r} - \frac{\mathbf{s}}{2}, t) \\
&= -i\hbar \int d^3r \{ (r[\nabla - \nabla'])_{\lambda\mu} \rho^\tau(\mathbf{r}, \mathbf{r}', t) \}_{r=r'} \\
&= -\frac{i\hbar}{2} \sum_{\sigma} \int d^3r \{ (r[\nabla - \nabla'])_{\lambda\mu} \rho(\mathbf{r}\sigma\tau, \mathbf{r}'\sigma\tau, t) \}_{r=r'} \\
&= -\frac{i\hbar}{2} \sum_{pq} \sum_{\sigma} \int d^3r \{ \phi_p^*(\mathbf{r}\sigma\tau) (r\nabla)_{\lambda\mu} \phi_q(\mathbf{r}\sigma\tau) \\
&\quad - \phi_q(\mathbf{r}\sigma\tau) (r\nabla)_{\lambda\mu} \phi_p^*(\mathbf{r}\sigma\tau) \} \langle \Psi | a_p^\dagger a_q | \Psi \rangle \\
&= \sum_{pq} \{ ((r\hat{p})_{\lambda\mu})_{pq}^\tau + i\hbar \frac{\sqrt{3}}{2} \delta_{\lambda 0} \sum_{\sigma} \int d^3r \phi_p^*(\mathbf{r}\sigma\tau) \phi_q(\mathbf{r}\sigma\tau) \} \rho_{qp}(t) \\
&= \langle \Psi | \sum_{s=1}^{N_\tau} (r_s \hat{p}_s)_{\lambda\mu} | \Psi \rangle + i\hbar \frac{\sqrt{3}}{2} \delta_{\lambda 0} N_\tau = \langle \Psi | \hat{L}_{\lambda\mu}^\tau | \Psi \rangle + i\hbar \frac{\sqrt{3}}{2} \delta_{\lambda 0} N_\tau. \tag{90}
\end{aligned}$$

The variation of this variable is

$$\begin{aligned}
\delta L_{\lambda\mu}^\tau(t) &= \mathcal{L}_{\lambda\mu}^\tau(t) = -i\hbar \int d^3r \{ (r[\nabla - \nabla'])_{\lambda\mu} \delta \rho^\tau(\mathbf{r}, \mathbf{r}', t) \}_{r=r'} \\
&= \sum_{pq} ((r\hat{p})_{\lambda\mu})_{pq}^\tau \rho_{qp}^{(1)}(t) \\
&= \sum_{\nu} [\langle 0 | \hat{L}_{\lambda\mu}^\tau | \nu \rangle c_{\nu} - \langle \nu | \hat{L}_{\lambda\mu}^\tau | 0 \rangle \bar{c}_{\nu}] e^{-i\Omega t} \\
&\quad + \sum_{\nu} [\langle \nu | \hat{L}_{\lambda\mu}^\tau | 0 \rangle c_{\nu}^* - \langle 0 | \hat{L}_{\lambda\mu}^\tau | \nu \rangle \bar{c}_{\nu}^*] e^{i\Omega t}. \tag{91}
\end{aligned}$$

The third variable is

$$P_{\lambda\mu}^\tau(t) = 2(2\pi\hbar)^{-3} \int d^3p \int d^3r p_{\lambda\mu}^2 f^\tau(\mathbf{r}, \mathbf{p}, t)$$

$$\begin{aligned}
&= \frac{2}{(2\pi\hbar)^3} \int d^3r \int d^3s \int d^3p p_{\lambda\mu}^2 \exp(-i\mathbf{p} \cdot \mathbf{s}/\hbar) \rho^\tau(\mathbf{r} + \frac{\mathbf{s}}{2}, \mathbf{r} - \frac{\mathbf{s}}{2}, t) \\
&= -\frac{\hbar^2}{2} \int d^3r \{(\nabla - \nabla')_{\lambda\mu}^2 \rho^\tau(\mathbf{r}, \mathbf{r}', t)\}_{r=r'} \\
&= -\frac{\hbar^2}{4} \sum_{\sigma} \int d^3r \{(\nabla - \nabla')_{\lambda\mu}^2 \rho(\mathbf{r}\sigma\tau, \mathbf{r}'\sigma\tau, t)\}_{r=r'} \\
&= -\frac{\hbar^2}{4} \sum_{pq} \sum_{\sigma} \int d^3r \{ \phi_p^*(\mathbf{r}\sigma\tau) \nabla_{\lambda\mu}^2 \phi_q(\mathbf{r}\sigma\tau) + \phi_q(\mathbf{r}\sigma\tau) \nabla_{\lambda\mu}^2 \phi_p^*(\mathbf{r}\sigma\tau) \\
&\quad - 2[\nabla \phi_q(\mathbf{r}\sigma\tau) \nabla \phi_p^*(\mathbf{r}\sigma\tau)]_{\lambda\mu} \} < \Psi | a_p^\dagger a_q | \Psi > \\
&= -\hbar^2 \sum_{pq} \sum_{\sigma} \int d^3r \phi_p^*(\mathbf{r}\sigma\tau) \nabla_{\lambda\mu}^2 \phi_q(\mathbf{r}\sigma\tau) < \Psi | a_p^\dagger a_q | \Psi > \\
&= \sum_{pq} (\hat{p}_{\lambda\mu}^2)_{pq}^\tau \rho_{qp}(t) = < \Psi | \sum_{s=1}^{N_\tau} \hat{p}_{\lambda\mu}^2(s) | \Psi > = < \Psi | \hat{P}_{\lambda\mu}^\tau | \Psi > . \tag{92}
\end{aligned}$$

The variation of this variable is

$$\begin{aligned}
\delta P_{\lambda\mu}^\tau(t) &= \mathcal{P}_{\lambda\mu}^\tau(t) = -\frac{\hbar^2}{2} \int d^3r \{(\nabla - \nabla')_{\lambda\mu}^2 \delta \rho^\tau(\mathbf{r}, \mathbf{r}', t)\}_{r=r'} \\
&= \sum_{pq} (\hat{p}_{\lambda\mu}^2)_{pq}^\tau \rho_{qp}^{(1)}(t) \\
&= \sum_{\nu} [< 0 | \hat{P}_{\lambda\mu}^\tau | \nu > c_\nu - < \nu | \hat{P}_{\lambda\mu}^\tau | 0 > \bar{c}_\nu] e^{-i\Omega t} \\
&\quad + \sum_{\nu} [< \nu | \hat{P}_{\lambda\mu}^\tau | 0 > c_\nu^* - < 0 | \hat{P}_{\lambda\mu}^\tau | \nu > \bar{c}_\nu^*] e^{i\Omega t} . \tag{93}
\end{aligned}$$

The structure of variables $\mathcal{R}_{\lambda\mu}$, $\mathcal{L}_{\lambda\mu}$, $\mathcal{P}_{\lambda\mu}$ (89,91,93) demonstrates in an obvious way the relation between the WFM method and RPA. One sees, for example, that the dynamical equations for the WFM variables $\mathcal{R}_{\lambda\mu}$ is a linear combination of the dynamical equations (74) for the transition matrix elements $< 0 | \hat{R}_{\lambda\mu} | \nu >$, the mixing coefficients c_n and \bar{c}_n being determined by the structure of the wave packet (83). Naturally, the same is true for the variables $\mathcal{L}_{\lambda\mu}$, $\mathcal{P}_{\lambda\mu}$. The dynamical equation for $< 0 | \hat{R}_{\lambda\mu} | \nu >$ is in turn, the linear combination of RPA equations (33) (or (32) in the case of arbitrary interaction) for the amplitudes X_{pq}, Y_{pq} , the mixing coefficients being particle-hole matrix elements of the operator $\hat{R}_{\lambda\mu}$.

As we see, there exist exact relations between the dynamical equations for the variables of the WFM method (moments) and the RPA dynamical equations for the amplitudes X_{pq}, Y_{pq} . One should note however, that these relations are exact only in our simplified model, because in general both methods, to be exact, have to operate within an infinite number of dynamical equations. In RPA one replaces the infinite number of particle-hole pairs of the shell model by the infinite number of phonons with the hope that the essential

part of physics is described by the small number of the lowest energy collective phonons and, consequently, one can neglect the rest of (infinite number) phonons. The coupling of the dynamical equations for X_{pq}, Y_{pq} , corresponding to different particle-hole pairs is realized by the matrix elements of the nucleon-nucleon interaction (see equations (32)). An analogous situation is observed in the WFM method, where the dynamical equations for Cartesian tensors of rank $n = 2$ are coupled (by the interaction terms in (5)) with dynamical equations for tensors of rank $n = 3$, these equations being coupled with the ones for tensors of rank $n = 4$ and so on up to $n = \infty$. And again one hopes that the essential part of physics is described by a few number of the lowest ranks tensors. This hope is based on the evident consideration that the higher rank tensors (moments) are responsible for more refined details and that, by neglecting them, one does not appreciably influence the description of the more global physics which is described with the lower ranks tensors. In this game of including only the lowest rank tensors one has to remember the trivial (but important) rule: the highest rank of tensors must not be less than the multipolarity of the studied motion.

It is easy to see that the nature of truncation in the two methods is quite different. So that in practical calculations with realistic Hamiltonians one can not establish the exact relation between these methods unless one works in the full space in both methods.

Of course there are exceptions like the case of the mean field potentials with quadratic coordinate dependence (harmonic oscillator with quadrupole-quadrupole or monopole-monopole residual interaction). Due to the huge degeneracy of the particle-hole configuration space all RPA sums are calculated analytically without any approximations. The same happens in WFM method – the dynamical equations for tensors of different ranks decouple and one obtains a finite set of equations, which can be solved exactly. As a consequence, both methods give identical results for integral characteristics of the collective motion, such as energies and transition probabilities.

A difference appears in the description of various distributions in coordinate space, for example, transition densities and currents, where the WFM method can not give the exact result, because it deals only with integrals over the whole phase space $\{\mathbf{p}, \mathbf{r}\}$. However, in principle the WFM method can give any number of moments required to construct approximate expressions for these distributions (see below).

4.1 Flows

We want to know the trajectories of infinitesimal displacements of neutrons and protons during their vibrational motion (the lines of currents). The infinitesimal displacements are determined by the magnitudes and directions of the nucleon velocities $\mathbf{u}(\mathbf{r}, t)$, given by

$$\begin{aligned}
m\rho(\mathbf{r}, t)\mathbf{u}(\mathbf{r}, t) &= \int \frac{4d^3p}{(2\pi\hbar)^3} \mathbf{p} f(\mathbf{r}, \mathbf{p}, t) \\
&= \frac{4}{(2\pi\hbar)^3} \int d^3s \int d^3p \mathbf{p} \exp(-i\mathbf{p} \cdot \mathbf{s}/\hbar) \rho(\mathbf{r} + \frac{\mathbf{s}}{2}, \mathbf{r} - \frac{\mathbf{s}}{2}, t) \\
&= -2i\hbar \{(\nabla - \nabla')\rho(\mathbf{r}, \mathbf{r}', t)\}_{r=r'} = -\frac{i\hbar}{2} \sum_{\sigma, \tau} \{(\nabla - \nabla')\rho(\mathbf{r}\sigma\tau, \mathbf{r}'\sigma\tau, t)\}_{r=r'} \\
&= -\frac{i\hbar}{2} \sum_{pq} \sum_{\sigma, \tau} \{\phi_p^*(\mathbf{r}\sigma\tau) \nabla \phi_q(\mathbf{r}\sigma\tau) - \phi_q(\mathbf{r}\sigma\tau) \nabla \phi_p^*(\mathbf{r}\sigma\tau)\} < \Psi | a_p^\dagger a_q | \Psi > \\
&= m \sum_{pq} j_{pq}(\mathbf{r}) \rho_{qp}(t) = m < \Psi | \sum_{pq} j_{pq}(\mathbf{r}) a_p^\dagger a_q | \Psi > \\
&= m < \Psi | \hat{J}(\mathbf{r}) | \Psi > .
\end{aligned} \tag{94}$$

The current density operator $\hat{J}(\mathbf{r})$ has the standard quantum mechanical definition [3]:

$$\begin{aligned}
\hat{J}(\mathbf{r}) &= \sum_{s=1}^A \hat{j}_s(\mathbf{r}) = \frac{\hbar}{2mi} \sum_{s=1}^A [\delta(\mathbf{r} - \hat{\mathbf{r}}_s) \nabla_s + \nabla_s \delta(\mathbf{r} - \hat{\mathbf{r}}_s)] = \sum_{pq} j_{pq}(\mathbf{r}) a_p^\dagger a_q, \\
j_{pq}(\mathbf{r}) &= < p | \frac{\hbar}{2mi} [\delta(\mathbf{r} - \hat{\mathbf{r}}) \nabla + \nabla \delta(\mathbf{r} - \hat{\mathbf{r}})] | q > \\
&= \sum_{\sigma, \tau} \frac{\hbar}{2mi} [\phi_p^*(\mathbf{r}\sigma\tau) \nabla \phi_q(\mathbf{r}\sigma\tau) - \phi_q(\mathbf{r}\sigma\tau) \nabla \phi_p^*(\mathbf{r}\sigma\tau)] = 4 \frac{\hbar}{2mi} [\phi_p^*(\mathbf{r}) \nabla \phi_q(\mathbf{r}) - \phi_q(\mathbf{r}) \nabla \phi_p^*(\mathbf{r})].
\end{aligned}$$

The variation of \mathbf{u} generated by the external field (82) is

$$\begin{aligned}
\rho^{eq}(\mathbf{r}) \delta \mathbf{u}(\mathbf{r}, t) &= \sum_{pq} j_{pq}(\mathbf{r}) \rho_{qp}^{(1)}(t) \\
&= \sum_{\nu} [< 0 | \hat{J}(\mathbf{r}) | \nu > c_{\nu} - < \nu | \hat{J}(\mathbf{r}) | 0 > \bar{c}_{\nu}] e^{-i\Omega t} \\
&\quad + \sum_{\nu} [< \nu | \hat{J}(\mathbf{r}) | 0 > c_{\nu}^* - < 0 | \hat{J}(\mathbf{r}) | \nu > \bar{c}_{\nu}^*] e^{i\Omega t}.
\end{aligned} \tag{95}$$

To proceed further two ways are possible.

The first, so to say direct way, is obvious. Having solutions (34) for X_{mi}^{ν} , Y_{mi}^{ν} we can calculate transition currents with the help of formula (47):

$$\begin{aligned}
< 0 | \hat{J}(\mathbf{r}) | \nu > &= \sum_{mi} (j_{im} X_{mi}^{\nu} + j_{mi} Y_{mi}^{\nu}) = K_{\nu} \sum_{mi} \left\{ \frac{j_{im} D_{im}^*}{E_{\nu} - \epsilon_{mi}} - \frac{j_{mi} D_{mi}^*}{E_{\nu} + \epsilon_{mi}} \right\} \\
&= K_{\nu} \left\{ \sum_{mi(\Delta N=0)} \left[\frac{j_{im} D_{im}^*}{E_{\nu} - \epsilon_0} - \frac{j_{mi} D_{mi}^*}{E_{\nu} + \epsilon_0} \right] + \sum_{mi(\Delta N=2)} \left[\frac{j_{im} D_{im}^*}{E_{\nu} - \epsilon_2} - \frac{j_{mi} D_{mi}^*}{E_{\nu} + \epsilon_2} \right] \right\}.
\end{aligned} \tag{96}$$

The operator D has a finite number of particle-hole matrix elements D_{mi} , so, in principle, the sums in (96) can be calculated exactly. The same is true for the coefficients c_ν . Therefore, one can find the exact (in the frame of RPA) result for the velocity distribution $\delta\mathbf{u}(\mathbf{r}, t)$. However, even in this simple model one can not find a compact analytical expression for sums in (96) – the field of velocities can be constructed only numerically.

The second way allows one to derive an approximate analytical expression for $\delta\mathbf{u}(\mathbf{r}, t)$. The main idea lies in the parametrization of the infinitesimal displacements $\xi(\mathbf{r}, t)$ [5]. Let us recall the main points. By definition $\delta\mathbf{u}_i(\mathbf{r}, t) = \frac{\partial \xi_i(\mathbf{r}, t)}{\partial t}$. The displacement ξ_i is parametrized [1] by the expansion

$$\xi_i(\mathbf{r}, t) = G_i(t) + \sum_{j=1}^3 G_{i,j}(t)x_j + \sum_{j,k=1}^3 G_{i,jk}(t)x_jx_k + \sum_{j,k,l=1}^3 G_{i,jkl}(t)x_jx_kx_l + \dots \quad (97)$$

which, in principle, is infinite, however one makes the approximation keeping only the first terms and neglecting all the rest of it. For example, in [1] only the two first terms were kept. It turned out, that $G_i = 0$ due to the triplanar symmetry of considered nuclei. The coefficients $G_{i,j}$ were expressed analytically in terms of the variables $\mathcal{R}_{21}(t)$ and $\mathcal{L}_{11}(t)$. Using the dynamical relations between $\mathcal{R}_{21}(t)$ and $\mathcal{L}_{11}(t)$ given by the last equation of the set (16), the final formulae for $\xi_i(\mathbf{r}, t)$ were found to be

$$\xi_1 = \sqrt{2}B\mathcal{J}_{13}x_3, \quad \xi_2 = \sqrt{2}B\mathcal{J}_{23}x_3, \quad \xi_3 = \sqrt{2}A(\mathcal{J}_{13}x_1 + \mathcal{J}_{23}x_2)$$

with

$$\begin{aligned} \mathcal{J}_{13} &= (\mathcal{R}_{2-1} - \mathcal{R}_{21})/2, \quad \mathcal{J}_{23} = i(\mathcal{R}_{2-1} + \mathcal{R}_{21})/2, \\ A &= \frac{3}{\sqrt{2}}[1 - 2\frac{\bar{\omega}^2}{\Omega^2}(1 - \alpha)\delta]/[Q_{00}(1 - \frac{2}{3}\delta)], \\ B &= \frac{3}{\sqrt{2}}[1 + 2\frac{\bar{\omega}^2}{\Omega^2}(1 - \alpha)\delta]/[Q_{00}(1 + \frac{4}{3}\delta)], \end{aligned} \quad (98)$$

The pole structure of the right hand side of equation (95) tells us, that the transition current can be calculated by means of an expression analogous to (21):

$$< 0|\hat{J}_i(\mathbf{r})|\nu > = \hbar \lim_{\Omega \rightarrow \Omega_\nu} (\Omega - \Omega_\nu)\rho^{eq}(\mathbf{r})\overline{\xi_i(\mathbf{r}, t) \exp(i\Omega t)} / < \nu|\hat{W}|0 > . \quad (99)$$

For the ξ_i from above we obtain (using formulae (89) and (84))

$$< 0|\hat{J}_1(\mathbf{r})|\nu > = -i\Omega_\nu\rho^{eq}(\mathbf{r})\frac{B}{\sqrt{2}} < 0|\hat{R}_{2-1} - \hat{R}_{21}|\nu > x_3,$$

$$\begin{aligned}
\langle 0|\hat{J}_2(\mathbf{r})|\nu \rangle &= \Omega_\nu \rho^{eq}(\mathbf{r}) \frac{B}{\sqrt{2}} \langle 0|\hat{R}_{2-1} + \hat{R}_{21}|\nu \rangle x_3, \\
\langle 0|\hat{J}_3(\mathbf{r})|\nu \rangle &= -i\Omega_\nu \rho^{eq}(\mathbf{r}) \frac{A}{\sqrt{2}} [\langle 0|\hat{R}_{2-1} - \hat{R}_{21}|\nu \rangle x_1 + i \langle 0|\hat{R}_{2-1} + \hat{R}_{21}|\nu \rangle x_2].
\end{aligned}$$

It is obvious that the second way is more adequate for the WFM method, because the moments $\mathcal{R}_{21}(t)$ and $\mathcal{L}_{11}(t)$ are just WFM variables and the dynamical relation between them is just given by the WFM dynamical equation.

If necessary, one can find the next term of the series (97). To calculate the respective coefficients $G_{i,jkl}(t)$ in the WFM method one is obliged to derive (and solve) the set of dynamical equations for higher (fourth) order moments of the Wigner function. To solve the same problem with RPA, it is necessary to construct the analogous set of dynamical equations for transition matrix elements of the respective operators. The required work is approximately of the same order of complexity in both cases.

In conclusion in full RPA one must calculate the currents numerically leading to fine details (shell effects) whereas in WFM and approximate RPA treatment one obtains their gross structure with analytical formulas. The latter feature is quite important in order to understand the real character of the motion under study since current patterns produced numerically from complicated formulas with a lot of summations like in (96) can hardly be interpreted physically. A good example is the interplay of the scissors mode and the isovector giant quadrupole resonance. Looking only at the flow patterns (see Figs. 1, 2 in [1]) one would not be able to tell that the former is mostly rotational with a small amount of an irrotational component and the other way round for the latter, as this can be seen from eqs. (42)–(47) in [1]. In this respect it is important to work with the infinitesimal displacements ξ_i , because by definition they are differentials ($\xi_1 = dx, \xi_2 = dy, \xi_3 = dz$) which allow one to construct differential equations for the current fields. For example equation (99), showing that transition current is proportional to a differential, allows one to derive a differential equation for the current field in RPA directly from (96). For example

$$\frac{\langle 0|\hat{J}_1(\mathbf{r})|\nu \rangle}{\langle 0|\hat{J}_2(\mathbf{r})|\nu \rangle} = \frac{dx}{dy}.$$

5 Conclusion

The properties of collective excitations (the scissors mode, isovector and isoscalar giant quadrupole resonances) of the harmonic oscillator Hamiltonian with the quadrupole–quadrupole residual interaction (HO+ QQ) have been studied with two methods: WFM and RPA. We have found that both methods give the same analytical expressions for energies and transition probabilities of all considered excitations. This, however, does not mean that WFM and RPA are identical approaches in all respects. For example current distributions are described differently in the two approaches even in this simple model. In general both methods are not equivalent unless one makes sure that the space of moments corresponds exactly to the particle–hole space used in RPA. However, the spirit of WFM is rather to drastically reduce the dimensions in considering only low order rank tensors. In this way, of course, one will lose the fine structure in the spectrum but still the gross structure will be well approximated. One also may check the convergence of the method in increasing the number of moments. In the case of well defined resonances only some more satellites to the main peak should appear. Such a method may be particularly useful in the case of deformed nuclei where the dimension of the RPA matrices becomes easily prohibitive.

It makes no sense to speak about advantages or disadvantages of one of the two discussed methods – they are complementary. Of course, RPA gives complete, exhaustive information concerning the microscopic (particle–hole) structure of collective excitations. However, sometimes considerable additional effort is required to understand their physical nature. On the contrary, WFM method gives direct information on the physical nature of the excitations. Our results serve as a very good illustration of this situation. What do we learn about the scissors mode and IVGQR from each of the two methods? RPA says that the scissors mode is mostly created by $\Delta N = 0$ particle–hole excitations with a small admixture of $\Delta N = 2$ particle–hole excitations and vice versa for IVGQR. Without further effort – this is about all. One does even not suspect the key role of the relative angular momentum in the creation of the scissors mode. On the other hand, the WFM method directly reveals that the scissors mode appears due to oscillations of the relative angular momentum with a small admixture of the quadrupole mode and vice versa for IVGQR. Further, it informs us about the extremely important role of the Fermi surface

deformation in the formation of the scissors mode. This demonstrates very well the difference between two approaches: the RPA describes the fine structure of collective excitations whereas the WFM method gives their gross structure.

Two new mathematical results are obtained for the HO+QQ model. We have proved exactly, without any approximations, the orthogonality of the "spurious" state to all physical states. In this sense, we have generalized the result of Lo Iudice [13] derived in a small deformation approximation. The analytical expressions are derived for the normalization factor of the synthetic scissors state and overlaps of this state with eigenstates of the model. It is shown, that the overlap of the synthetic scissors with the real scissors reaches its maximal value ~ 0.6 in a small deformation limit. The increasing of δ and /or taking into account pairing correlations decreases the overlap, that is confirmed by numerous microscopic calculations with various forces [17]. Such small overlap leads inevitably to the conclusion, that the original model of counter rotating rigid rotors [10] has not very much in common with the real scissors mode, the correct description of which requires the proper treatment of the Fermi surface deformation and the coupling with IVGQR.

Appendix A

It is known that the deformed harmonic oscillator Hamiltonian can be obtained in a Hartree approximation "by making the assumption that the isoscalar part of the QQ force builds the one-body container well" [15]. In our case it is obtained quite easily by summing the expressions for V^p and V^n (formula (7)):

$$V(\mathbf{r}, t) = \frac{1}{2}(V^p(\mathbf{r}, t) + V^n(\mathbf{r}, t)) = \frac{1}{2}m\omega^2 r^2 + \kappa_0 \sum_{\mu=-2}^2 (-1)^\mu Q_{2-\mu}(t) q_{2\mu}(\mathbf{r}). \quad (\text{A.1})$$

In the state of equilibrium (i.e., in the absence of an external field) $Q_{2\pm 1} = Q_{2\pm 2} = 0$. Using the definition [16] $Q_{20} = Q_{00}\frac{4}{3}\delta$ and the formula $q_{20} = 2z^2 - x^2 - y^2$ we obtain the potential of the anisotropic harmonic oscillator

$$V(\mathbf{r}) = \frac{m}{2}[\omega_x^2(x^2 + y^2) + \omega_z^2 z^2]$$

with oscillator frequencies

$$\omega_x^2 = \omega_y^2 = \omega^2(1 + \sigma\delta), \quad \omega_z^2 = \omega^2(1 - 2\sigma\delta),$$

where $\sigma = -\kappa_0 \frac{8Q_{00}}{3m\omega^2}$. The definition of the deformation parameter δ must be reproduced by the harmonic oscillator wave functions, which allows one to fix the value of σ . We have

$$Q_{00} = \frac{\hbar}{m} \left(\frac{\Sigma_x}{\omega_x} + \frac{\Sigma_y}{\omega_y} + \frac{\Sigma_z}{\omega_z} \right), \quad Q_{20} = 2 \frac{\hbar}{m} \left(\frac{\Sigma_z}{\omega_z} - \frac{\Sigma_x}{\omega_x} \right),$$

where $\Sigma_x = \Sigma_{i=1}^A (n_x + \frac{1}{2})_i$ and n_x is the oscillator quantum number. Using the self-consistency condition [16]

$$\Sigma_x \omega_x = \Sigma_y \omega_y = \Sigma_z \omega_z = \Sigma_0 \omega_0,$$

where Σ_0 and ω_0 are defined in the spherical case, we get

$$\frac{Q_{20}}{Q_{00}} = 2 \frac{\omega_x^2 - \omega_z^2}{\omega_x^2 + 2\omega_z^2} = \frac{2\sigma\delta}{1 - \sigma\delta} = \frac{4}{3}\delta.$$

Solving the last equation with respect to σ , we find

$$\sigma = \frac{2}{3 + 2\delta}. \quad (\text{A.2})$$

Therefore, the oscillator frequencies and the strength constant can be written as

$$\omega_x^2 = \omega_y^2 = \bar{\omega}^2 (1 + \frac{4}{3}\delta), \quad \omega_z^2 = \bar{\omega}^2 (1 - \frac{2}{3}\delta), \quad \kappa_0 = -\frac{m\bar{\omega}^2}{4Q_{00}} \quad (\text{A.3})$$

with $\bar{\omega}^2 = \omega^2 / (1 + \frac{2}{3}\delta)$. The condition for volume conservation $\omega_x \omega_y \omega_z = \text{const} = \omega_0^3$ makes ω δ -dependent

$$\omega^2 = \omega_0^2 \frac{1 + \frac{2}{3}\delta}{(1 + \frac{4}{3}\delta)^{2/3} (1 - \frac{2}{3}\delta)^{1/3}}.$$

So the final expressions for oscillator frequencies are

$$\omega_x^2 = \omega_y^2 = \omega_0^2 \left(\frac{1 + \frac{4}{3}\delta}{1 - \frac{2}{3}\delta} \right)^{1/3}, \quad \omega_z^2 = \omega_0^2 \left(\frac{1 - \frac{2}{3}\delta}{1 + \frac{4}{3}\delta} \right)^{2/3}. \quad (\text{A.4})$$

It is interesting to compare these expressions with the very popular [16, 3] parametrization

$$\omega_x^2 = \omega_y^2 = \omega'^2 (1 + \frac{2}{3}\delta'), \quad \omega_z^2 = \omega'^2 (1 - \frac{4}{3}\delta').$$

The volume conservation condition gives

$$\omega'^2 = \frac{\omega_0^2}{(1 + \frac{2}{3}\delta')^{2/3} (1 - \frac{4}{3}\delta')^{1/3}},$$

so the final expressions for oscillator frequencies are

$$\omega_x^2 = \omega_y^2 = \omega_0^2 \left(\frac{1 + \frac{2}{3}\delta'}{1 - \frac{4}{3}\delta'} \right)^{1/3}, \quad \omega_z^2 = \omega_0^2 \left(\frac{1 - \frac{4}{3}\delta'}{1 + \frac{2}{3}\delta'} \right)^{2/3}. \quad (\text{A.5})$$

The direct comparison of expressions (A.4) and (A.5) allows one to establish the following relation between δ and δ' :

$$\delta' = \frac{\delta}{1 + 2\delta}, \quad \delta = \frac{\delta'}{1 - 2\delta'}.$$

One more parametrization of oscillator frequencies can be found in the review [17]:

$$\omega_x^2 = \omega_y^2 = \frac{\omega''^2}{1 - \frac{2}{3}\delta''}, \quad \omega_z^2 = \frac{\omega''^2}{1 + \frac{4}{3}\delta''}.$$

One has from the volume conservation condition

$$\omega''^2 = \omega_0^2 \left(1 - \frac{2}{3}\delta''\right)^{2/3} \left(1 + \frac{4}{3}\delta''\right)^{1/3},$$

so the final expressions for oscillator frequencies are

$$\omega_x^2 = \omega_y^2 = \omega_0^2 \left(\frac{1 + \frac{4}{3}\delta''}{1 - \frac{2}{3}\delta''} \right)^{1/3}, \quad \omega_z^2 = \omega_0^2 \left(\frac{1 - \frac{2}{3}\delta''}{1 + \frac{4}{3}\delta''} \right)^{2/3}, \quad (\text{A.6})$$

that coincide exactly with (A.4), i.e. $\delta'' = \delta$.

It is easy to see that equations (A.4) correspond to the case when the deformed density $\rho(\mathbf{r})$ is obtained from the spherical density $\rho_0(r)$ by the scale transformation [9]

$$(x, y, z) \rightarrow (xe^{\alpha/2}, ye^{\alpha/2}, ze^{-\alpha})$$

with

$$e^\alpha = \left(\frac{1 + \frac{4}{3}\delta}{1 - \frac{2}{3}\delta} \right)^{1/3}, \quad \delta = \frac{3e^{3\alpha} - 1}{2e^{3\alpha} + 2}, \quad (\text{A.7})$$

which conserves the volume and does not destroy the self-consistency, because the density and potential are transformed in the same way.

It is necessary to note that Q_{00} also depends on δ

$$Q_{00} = \frac{\hbar}{m} \left(\frac{\Sigma_x}{\omega_x} + \frac{\Sigma_y}{\omega_y} + \frac{\Sigma_z}{\omega_z} \right) = \frac{\hbar}{m} \Sigma_0 \omega_0 \left(\frac{2}{\omega_x^2} + \frac{1}{\omega_z^2} \right) = Q_{00}^0 \frac{1}{(1 + \frac{4}{3}\delta)^{1/3} (1 - \frac{2}{3}\delta)^{2/3}},$$

where $Q_{00}^0 = A \frac{3}{5} R^2$, $R = r_0 A^{1/3}$. As a result, the final expression for the strength constant becomes

$$\kappa_0 = -\frac{m\omega_0^2}{4Q_{00}^0} \left(\frac{1 - \frac{2}{3}\delta}{1 + \frac{4}{3}\delta} \right)^{1/3} = -\frac{m\omega_0^2}{4Q_{00}^0} e^{-\alpha},$$

that coincides with the respective result of [9].

Appendix B

To calculate the sums $D_0 = \sum_{mi(\Delta N=0)} |D_{mi}|^2$ and $D_2 = \sum_{mi(\Delta N=2)} |D_{mi}|^2$ we employ the sum-rule techniques of Suzuki and Rowe [9]. The well known harmonic oscillator relations

$$\begin{aligned} x\psi_{n_x} &= \sqrt{\frac{\hbar}{2m\omega_x}}(\sqrt{n_x}\psi_{n_x-1} + \sqrt{n_x+1}\psi_{n_x+1}), \\ \hat{p}_x\psi_{n_x} &= -i\sqrt{\frac{m\hbar\omega_x}{2}}(\sqrt{n_x}\psi_{n_x-1} - \sqrt{n_x+1}\psi_{n_x+1}) \end{aligned} \quad (\text{B.1})$$

allow us to write

$$\begin{aligned} xz\psi_{n_x}\psi_{n_z} &= \frac{\hbar}{2m\sqrt{\omega_x\omega_z}}(\sqrt{n_x n_z}\psi_{n_x-1}\psi_{n_z-1} + \sqrt{(n_x+1)(n_z+1)}\psi_{n_x+1}\psi_{n_z+1} \\ &\quad + \sqrt{(n_x+1)n_z}\psi_{n_x+1}\psi_{n_z-1} + \sqrt{n_x(n_z+1)}\psi_{n_x-1}\psi_{n_z+1}), \\ \frac{\hat{p}_x\hat{p}_z}{m^2\omega_x\omega_z}\psi_{n_x}\psi_{n_z} &= -\frac{\hbar}{2m\sqrt{\omega_x\omega_z}}(\sqrt{n_x n_z}\psi_{n_x-1}\psi_{n_z-1} + \sqrt{(n_x+1)(n_z+1)}\psi_{n_x+1}\psi_{n_z+1} \\ &\quad - \sqrt{(n_x+1)n_z}\psi_{n_x+1}\psi_{n_z-1} - \sqrt{n_x(n_z+1)}\psi_{n_x-1}\psi_{n_z+1}). \end{aligned} \quad (\text{B.2})$$

These formulae demonstrate in an obvious way that the operators

$$P_0 = \frac{1}{2}(zx + \frac{1}{m^2\omega_x\omega_z}\hat{p}_x\hat{p}_z) \quad \text{and} \quad P_2 = \frac{1}{2}(zx - \frac{1}{m^2\omega_x\omega_z}\hat{p}_x\hat{p}_z)$$

contribute only to the excitation of the $\Delta N = 0$ and $\Delta N = 2$ states, respectively. Following [9], we express the zx component of $r^2 Y_{21} = \sqrt{\frac{5}{16\pi}}D = -\sqrt{\frac{15}{8\pi}}z(x + iy)$ as

$$zx = P_0 + P_2.$$

Hence, we have

$$\begin{aligned} \epsilon_0 \sum_{mi(\Delta N=0)} |<0|\sum_{s=1}^A z_s x_s |mi>|^2 &= \epsilon_0 \sum_{mi} |<0|\sum_{s=1}^A P_0(s) |mi>|^2 \\ &= \frac{1}{2} <0|[\sum_{s=1}^A P_0(s), [H, \sum_{s=1}^A P_0(s)]]|0>, \end{aligned} \quad (\text{B.3})$$

where $\epsilon_0 = \hbar(\omega_x - \omega_z)$. The above commutator is easily evaluated for the Hamiltonian with the potential (A.1), as

$$<0|[\sum_{s=1}^A P_0(s), [H, \sum_{s=1}^A P_0(s)]]|0> = \frac{\hbar}{2m}\epsilon_0 \left(\frac{<0|\sum_{s=1}^A z_s^2|0>}{\omega_x} - \frac{<0|\sum_{s=1}^A x_s^2|0>}{\omega_z} \right). \quad (\text{B.4})$$

Taking into account the axial symmetry and using the definitions

$$Q_{00} = \langle 0 | \sum_{s=1}^A (2x_s^2 + z_s^2) | 0 \rangle, \quad Q_{20} = 2 \langle 0 | \sum_{s=1}^A (z_s^2 - x_s^2) | 0 \rangle, \quad Q_{20} = Q_{00} \frac{4}{3} \delta,$$

we transform this expression to

$$\langle 0 | [\sum_{s=1}^A P_0(s), [H, \sum_{s=1}^A P_0(s)]] | 0 \rangle = \frac{\hbar}{6m} \epsilon_0 Q_{00} \left(\frac{1 + \frac{4}{3}\delta}{\omega_x} - \frac{1 - \frac{2}{3}\delta}{\omega_z} \right). \quad (\text{B.5})$$

With the help of the self-consistent expressions for ω_x , ω_z (A.3) one comes to the following result:

$$\langle 0 | [\sum_{s=1}^A P_0(s), [H, \sum_{s=1}^A P_0(s)]] | 0 \rangle = \frac{Q_{00}}{6m} \frac{\epsilon_0^2}{\bar{\omega}^2} = \frac{\hbar^2}{6m} Q_{00}^0 \left(\frac{\omega_0}{\omega_z} - \frac{\omega_0}{\omega_x} \right)^2. \quad (\text{B.6})$$

By using the fact that the matrix elements for the zy component of $r^2 Y_{21}$ are identical to those for the zx component, because of axial symmetry, we finally obtain

$$\epsilon_0 \sum_{mi(\Delta N=0)} |\langle 0 | \sum_{s=1}^A r_s^2 Y_{21} | mi \rangle|^2 = \frac{5}{16\pi} \frac{Q_{00}}{m \bar{\omega}^2} \epsilon_0^2 = \frac{5}{16\pi} \frac{Q_{00}^0}{m} \frac{\epsilon_0^2}{\omega_0^2} \left(\frac{1 + \frac{4}{3}\delta}{1 - \frac{2}{3}\delta} \right)^{1/3}. \quad (\text{B.7})$$

By calculating a double commutator for the P_2 operator, we find

$$\epsilon_2 \sum_{mi(\Delta N=2)} |\langle 0 | \sum_{s=1}^A r_s^2 Y_{21} | mi \rangle|^2 = \frac{5}{16\pi} \frac{Q_{00}}{m \bar{\omega}^2} \epsilon_2^2 = \frac{5}{16\pi} \frac{Q_{00}^0}{m} \frac{\epsilon_2^2}{\omega_0^2} \left(\frac{1 + \frac{4}{3}\delta}{1 - \frac{2}{3}\delta} \right)^{1/3}, \quad (\text{B.8})$$

where $\epsilon_2 = \hbar(\omega_x + \omega_z)$.

We need also the sums D_0^τ and D_2^τ calculated separately for neutron and proton systems with the mean fields V^n and V^p , respectively. The necessary formulae are easily derivable from the already obtained results. There are no any reasons to require the fulfillment of the self-consistency conditions for neutrons and protons separately, so one has to use formula (B.5). The trivial change of notation gives

$$\langle 0 | [\sum_{s=1}^Z P_0(s), [H^p, \sum_{s=1}^Z P_0(s)]] | 0 \rangle = \frac{\hbar}{6m} \epsilon_0^p Q_{00}^p \left(\frac{1 + \frac{4}{3}\delta^p}{\omega_x^p} - \frac{1 - \frac{2}{3}\delta^p}{\omega_z^p} \right), \quad (\text{B.9})$$

$$\epsilon_0^p \sum_{mi(\Delta N=0)} |\langle 0 | \sum_{s=1}^Z r_s^2 Y_{21} | mi \rangle|^2 = \frac{5}{16\pi} \frac{\hbar}{m} \epsilon_0^p Q_{00}^p \left(\frac{1 + \frac{4}{3}\delta^p}{\omega_x^p} - \frac{1 - \frac{2}{3}\delta^p}{\omega_z^p} \right), \quad (\text{B.10})$$

$$\epsilon_2^p \sum_{mi(\Delta N=2)} |\langle 0 | \sum_{s=1}^Z r_s^2 Y_{21} | mi \rangle|^2 = \frac{5}{16\pi} \frac{\hbar}{m} \epsilon_2^p Q_{00}^p \left(\frac{1 + \frac{4}{3}\delta^p}{\omega_x^p} + \frac{1 - \frac{2}{3}\delta^p}{\omega_z^p} \right). \quad (\text{B.11})$$

The nontrivial information is contained in oscillator frequencies of the mean fields V^p and V^n (formula (7))

$$\begin{aligned}(\omega_x^p)^2 &= \omega^2[1 - \frac{2}{m\omega^2}(\kappa Q_{20}^p + \bar{\kappa} Q_{20}^n)], & (\omega_z^p)^2 &= \omega^2[1 + \frac{4}{m\omega^2}(\kappa Q_{20}^p + \bar{\kappa} Q_{20}^n)], \\(\omega_x^n)^2 &= \omega^2[1 - \frac{2}{m\omega^2}(\kappa Q_{20}^n + \bar{\kappa} Q_{20}^p)], & (\omega_z^n)^2 &= \omega^2[1 + \frac{4}{m\omega^2}(\kappa Q_{20}^n + \bar{\kappa} Q_{20}^p)].\end{aligned}\quad (\text{B.12})$$

The above-written formulae can be used also to calculate the analogous sums for operators containing various combinations of momenta and coordinates, for example, components of an angular momentum, tensor products $(r\hat{p})_{21}$ and $(\hat{p}^2)_{21}$. Really, by definition $\hat{I}_1 = y\hat{p}_z - z\hat{p}_y$, $\hat{I}_2 = z\hat{p}_x - x\hat{p}_z$. In accordance with (B.1), we have

$$\begin{aligned}x\hat{p}_z\psi_{n_x}\psi_{n_z} &= -i\frac{\hbar}{2}\sqrt{\frac{\omega_z}{\omega_x}}(\sqrt{n_x n_z}\psi_{n_x-1}\psi_{n_z-1} - \sqrt{(n_x+1)(n_z+1)}\psi_{n_x+1}\psi_{n_z+1} \\&\quad + \sqrt{(n_x+1)n_z}\psi_{n_x+1}\psi_{n_z-1} - \sqrt{n_x(n_z+1)}\psi_{n_x-1}\psi_{n_z+1}).\end{aligned}\quad (\text{B.13})$$

Therefore,

$$\begin{aligned}\hat{I}_2\psi_{n_x}\psi_{n_z} &= i\frac{\hbar}{2}(\sqrt{\frac{\omega_z}{\omega_x}} - \sqrt{\frac{\omega_x}{\omega_z}})(\sqrt{n_x n_z}\psi_{n_x-1}\psi_{n_z-1} - \sqrt{(n_x+1)(n_z+1)}\psi_{n_x+1}\psi_{n_z+1}) \\&\quad + i\frac{\hbar}{2}(\sqrt{\frac{\omega_z}{\omega_x}} + \sqrt{\frac{\omega_x}{\omega_z}})(\sqrt{(n_x+1)n_z}\psi_{n_x+1}\psi_{n_z-1} - \sqrt{n_x(n_z+1)}\psi_{n_x-1}\psi_{n_z+1}).\end{aligned}\quad (\text{B.14})$$

Having formulae (B.2) and (B.14), one derives the following expressions for matrix elements coupling the ground state with $\Delta N = 2$ and $\Delta N = 0$ excitations:

$$\begin{aligned}\langle n_x+1, n_z+1|\hat{I}_2|0\rangle &= i\frac{\hbar}{2}\frac{(\omega_x^2 - \omega_z^2)}{\omega_x + \omega_z}\sqrt{\frac{(n_x+1)(n_z+1)}{\omega_x\omega_z}}, \\ \langle n_x+1, n_z-1|\hat{I}_2|0\rangle &= i\frac{\hbar}{2}\frac{(\omega_x^2 - \omega_z^2)}{\omega_x - \omega_z}\sqrt{\frac{(n_x+1)n_z}{\omega_x\omega_z}}, \\ \langle n_x+1, n_z+1|xz|0\rangle &= \frac{\hbar}{2m}\sqrt{\frac{(n_x+1)(n_z+1)}{\omega_x\omega_z}}, \\ \langle n_x+1, n_z-1|xz|0\rangle &= \frac{\hbar}{2m}\sqrt{\frac{(n_x+1)n_z}{\omega_x\omega_z}}.\end{aligned}\quad (\text{B.15})$$

It is easy to see that

$$\begin{aligned}\langle n_x+1, n_z+1|\hat{I}_2|0\rangle &= im\frac{(\omega_x^2 - \omega_z^2)}{\omega_x + \omega_z} \langle n_x+1, n_z+1|xz|0\rangle, \\ \langle n_x+1, n_z-1|\hat{I}_2|0\rangle &= im\frac{(\omega_x^2 - \omega_z^2)}{\omega_x - \omega_z} \langle n_x+1, n_z-1|xz|0\rangle.\end{aligned}$$

Due to the degeneracy of the model all particle-hole excitations with $\Delta N = 2$ have the same energy ϵ_2 and all particle-hole excitations with $\Delta N = 0$ have the energy ϵ_0 . This fact allows one to join the last two formulae into one general expression

$$\langle ph|\hat{I}_2|0 \rangle = i\hbar m \frac{(\omega_x^2 - \omega_z^2)}{\epsilon_{ph}} \langle ph|xz|0 \rangle.$$

Taking into account the axial symmetry we can write the analogous formula for \hat{I}_1 :

$$\langle ph|\hat{I}_1|0 \rangle = -i\hbar m \frac{(\omega_x^2 - \omega_z^2)}{\epsilon_{ph}} \langle ph|yz|0 \rangle.$$

The magnetic transition operator (22) is proportional to the angular momentum: $\hat{f}_{1\pm 1} = -\frac{ie}{4mc} \sqrt{\frac{3}{2\pi}} (\hat{I}_2 \mp i\hat{I}_1)$ Therefore, we can write

$$\langle ph|\hat{f}_{1\pm 1}|0 \rangle = -\frac{e\hbar}{2c\sqrt{5}} \frac{(\omega_x^2 - \omega_z^2)}{\epsilon_{ph}} \langle ph|r^2 Y_{2\pm 1}|0 \rangle. \quad (\text{B.16})$$

Similar calculations for the tensor product $(r\hat{p})_{21} = -\frac{1}{2}[z\hat{p}_x + x\hat{p}_z + i(z\hat{p}_y + y\hat{p}_z)]$ lead to the following relation:

$$\langle ph|(r\hat{p})_{21}|0 \rangle = i\frac{m}{\hbar} \sqrt{\frac{2\pi}{15}} \epsilon_{ph} \langle ph|r^2 Y_{2\pm 1}|0 \rangle = i\frac{m}{2\hbar} \epsilon_{ph} \langle ph|r_{21}^2|0 \rangle. \quad (\text{B.17})$$

Two kinds of particle-hole matrix elements are obtained from the second formula of (B.2):

$$\begin{aligned} \langle n_x + 1, n_z + 1|\hat{p}_x \hat{p}_z|0 \rangle &= -\hbar m \omega_x \omega_z \sqrt{\frac{(n_x + 1)(n_z + 1)}{2\omega_x 2\omega_z}}, \\ \langle n_x + 1, n_z - 1|\hat{p}_x \hat{p}_z|0 \rangle &= \hbar m \omega_x \omega_z \sqrt{\frac{(n_x + 1)n_z}{2\omega_x 2\omega_z}}. \end{aligned}$$

Simple comparison with (B.15) shows that

$$\begin{aligned} \langle n_x + 1, n_z + 1|\hat{p}_x \hat{p}_z|0 \rangle &= -m^2 \omega_x \omega_z \langle n_x + 1, n_z + 1|xz|0 \rangle, \\ \langle n_x + 1, n_z - 1|\hat{p}_x \hat{p}_z|0 \rangle &= m^2 \omega_x \omega_z \langle n_x + 1, n_z - 1|xz|0 \rangle. \end{aligned}$$

With the help of obvious relations

$$2\omega_x \omega_z = \omega_x^2 + \omega_z^2 - \epsilon_0^2/\hbar^2, \quad -2\omega_x \omega_z = \omega_x^2 + \omega_z^2 - \epsilon_2^2/\hbar^2$$

these two formulae can be joined into one expression

$$\langle ph|\hat{p}_x \hat{p}_z|0 \rangle = \frac{m^2}{2} (\omega_x^2 + \omega_z^2 - \epsilon_{ph}^2/\hbar^2) \langle ph|xz|0 \rangle.$$

By definition $\hat{p}_{21}^2 = -\hat{p}_z(\hat{p}_x + i\hat{p}_y)$ and $\hat{r}_{21}^2 = -z(x + iy)$, hence,

$$\langle ph|\hat{p}_{21}^2|0 \rangle = \frac{m^2}{2} (\omega_x^2 + \omega_z^2 - \epsilon_{ph}^2/\hbar^2) \langle ph|r_{21}^2|0 \rangle. \quad (\text{B.18})$$

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